Sequence weakly converging to 0 but not strongly convergent

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Exercise 3.1.6

In the Hilbert space $l^2(\mathbb{N})$, consider the sequence $(f_j)_{j\in\mathbb{N}}$ given by $f_j(k) = \delta_{jk}$, meaning $f_j(k) = 1$ if $j = k$ and $f_j(k) = 0$ if $j \neq k$. Show that this sequence weakly converges to 0 but does not converge strongly. More generally, let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of an infinite dimensional Hilbert space. Show that $w - \lim_{j\to\infty} e_j = 0$, but that $s - \lim_{j\to\infty} e_j$ does not exist.

1. For sequence $(f_j)_{j\in\mathbb{N}} \subset \mathcal{H}$, $\forall g \in \mathcal{H} = l^2(\mathbb{N})$, $\sum_{k\in\mathbb{N}} |g(k)|^2 < \infty$, and so we have

$$\lim_{k\to\infty} g(k) = 0.$$

The scalar product of g and f_j satisfies

$$\langle g, f_j \rangle = g(j)$$

since $f_j(k) = 1$ if $j = k$ and $f_j(k) = 0$ otherwise.
Therefore, it is derived that

$$\lim_{j\to\infty} \langle g, f_j \rangle = \lim_{j\to\infty} g(j) = 0.$$

The property of weak convergence is then proved.

Obviously $\|f_j\| = 1$, $\lim_{j\in\mathbb{N}} \|f_j\| = 1 \neq 0$, the sequence does not converge strongly to 0.

2. For the orthonormal basis, $\forall g \in \mathcal{H}$,

$$\|g\|^2 = \sum_{j \in \mathbb{N}} |\langle g, e_j \rangle|^2.$$

Since $\|g\|^2$ is finite and $|\langle g, e_j \rangle|^2 > 0$, we can infer

$$\lim_{j \to \infty} \langle g, e_j \rangle = 0.$$

$w - \lim_{j \to \infty} e_j = 0$ is proved.

Strongly converging to 0 is impossible since $\forall j \in \mathbb{N}$, $\|e_j\| = 1$.

3. Actually neither of the two sequences is strongly convergent to anything, because if it is, $s - \lim_{j \to \infty} f_j = f \neq 0$ as an example, according to Lemma 3.1.7, we get

$$w - \lim_{j \to \infty} f_j = f$$

contradictory to what we have proved $w - \lim_{j \to \infty} f_j = 0$.

This exercise gives an instruction that weak convergence does not lead necessarily to strong convergence.

Reference

Amrein, W 2009, *Hilbert Space Methods in Quantum Mechanics*, EPFL Press, Laussane.