Proof Lemma 2.6.9

Lemma 2.6.9. *Let $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$. Then for any $a, b \geq 0$ one has*

$$ab \leq \alpha a^{\frac{1}{\alpha}} + \beta b^{\frac{1}{\beta}}.$$

To proof this inequality, I will use Jensen's inequality.

Jensen's inequality

For any $n \in \mathbb{N}$, with $\sum_{i=1}^{n} t_i = 1$, $t_i \geq 0$
a convex f satisfies

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i)$$

* definition of convex function
$f: [a,b] \to \mathbb{R}$ is convex if and only if a following condition holds:
For all $t \in [0,1]$ and all $x_1, x_2 \in [a,b]$

$$f(t x_2 + (1-t) x_1) \leq t f(x_1) + (1-t) f(x_2)$$

Proof of Jensen's inequality

Let me proof by induction.

Bace: if $n=1$ then $t_1 = 1$ so the inequality is $f(x_1) \leq f(x_1)$, which is obiously true.

if $n=2$, the inequality is $f(t_1 x_1 + t_2 x_2) \leq t_1 f(x_1) + t_2 f(x_2)$, which is true by the convexity of f.

Inductive: I will prove if the inequality holds for $n = k \geq 2$ then it also holds for $n = k+1$.

if $n = k+1$, the left side:

$$f\left(\sum_{i=1}^{k+1} t_i x_i\right) = f\left(\sum_{i=1}^{k} t_i x_i + t_{k+1} x_{k+1}\right)$$

Let me set $1 - t_{k+1} = a$

If $a = 0$ ($t_{k+1} = 1$), then other $t_i = 0$, so the inequality is $f(x_{k+1}) \leq f(x_{k+1})$, which is true.

If $a \neq 0$, since $a + t_{k+1} = 1$ and f is a convex function,

$$f\left(\sum_{i=1}^{k+1} t_i x_i\right) = f\left(a \cdot \sum_{i=1}^{k} \frac{t_i}{a} x_i + t_{k+1} x_{k+1}\right)$$

$$\leq a \cdot f\left(\sum_{i=1}^{k} \frac{t_i}{a} x_i\right) + t_{k+1} f(x_{k+1}) \quad \cdots (*)$$

also, since $\sum_{i=1}^{k} \frac{t_i}{a} = \frac{1 - t_{k+1}}{1 - t_{k+1}} = 1$, by the inductive hypothesis,

$$(*) \leq a \left\{\sum_{i=1}^{k} \frac{t_i}{a} f(x_i)\right\} + t_{k+1} f(x_{k+1})$$

$$= \sum_{i=1}^{k+1} t_k f(x_i) \quad \square$$

Now, I can apply Jensen's inequality to prove the Lemma.

Set $f(x) = e^x$ f is convex since $\frac{d^2}{dx^2} f = e^x > 0$

Then for $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$, the following inequality holds for all x_1, x_2:

$$e^{\alpha x_1 + \beta x_2} \leq \alpha e^{x_1} + \beta e^{x_2}$$

Set $x_1 = \frac{1}{\alpha} \log a$, $x_2 = \frac{1}{\beta} \log b$ $(a, b \geq 0)$, I can get

$$ab \leq \alpha e^{\frac{1}{\alpha} \log a} + \beta e^{\frac{1}{\beta} \log b}$$

$$= \alpha a^{\frac{1}{\alpha}} + \beta b^{\frac{1}{\beta}} \quad \square$$

Ref. 美しき不等式の世界 佐藤淳郎(訳)