The Interior and Closure of a Set Change with the Norm

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In this report, we let $E = C([0,1], \mathbb{R})$ and $D = \{f \in E \mid f(0) = f(1)\}$. We equip E with the infinity norm and the L^1 norm respectively and try to specify $\overline{D}$ and D^o. Using this example, we find that the interior and closure of one set will be different if we change the norm.

The infinity norm

Solution

Let $f \in \overline{D}$, we have $|f(0) - f_n(0)| \leq \|f - f_n\|_\infty$. As $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$, hence $f(0) = \lim_{n\to\infty} f_n(0)$ and similarly $f(1) = \lim_{n\to\infty} f_n(1)$. As $f_n(0) = f_n(1)$ for all n, we have then $f(0) = f(1)$. Hence $f \in D$. The set D is therefore closed, $\overline{D} = D$.

For $f \in D^o$, by definition, $f \in E$ and there exists $r > 0$ such that $BO(f, r) \subset D^o \subset D$.

Now we set $g : x \mapsto f(x) - \frac{rx}{2}$ which satisfies that $\|f - g\|_\infty < r$. However, as $g(1) - g(0) = -\frac{r}{2} \neq 0$, we obtain that $g \notin D^\circ$. Thus D has an empty interior, $D^\circ = \emptyset$.

The L^1 norm

Solution

Let's equip E with the L^1 norm and determine $\overline{D}$ and D^o. We will show that $\overline{D} = E$. For $f \in E$, we will prove that there exists a sequence $(f_n)_{n\in\mathbb{N}^*} \subset D$ such that $\|f_n - f\|_{1,[0,1]} \to 0$. Let's define:

$$f_n(x) = \begin{cases} f(1) + n \times \left(f(\frac{1}{n}) - f(1) \right) \times x & \text{if } x \in \left[0, \frac{1}{n}\right] \\ f(x) & \text{if } x \in \left(\frac{1}{n}, 1\right] \end{cases}$$

For all $n \in \mathbb{N}^*$, $f_n \in E$ and $f_n(0) = f_n(1) = f(1)$. Therefore, $(f_n)_{n\in\mathbb{N}^*} \subset D$. Furthermore, since f is continuous on $[0,1]$, it is bounded (there exists $M \geq 0$ such that $|f(t)| \leq M$ for all $t \in [0,1]$). Using the triangle inequality, we have:

$$\|f_n - f\|_{1,[0,1]} \leq \int_0^{\frac{1}{n}} \left| f(x) - f(1) - n \times \left(f\left(\frac{1}{n}\right) - f(1) \right) \times x \right| dx \leq \frac{2M}{n} + \frac{2M}{2n} \leq \frac{3M}{n}$$

We have $\|f_n - f\|_{1,[0,1]} \to 0$ as $n \to +\infty$. Thus, $f \in \overline{D}$, implying $E \subset \overline{D}$. The other inclusion is clear, therefore $\overline{D} = E$. We note that for the L^1 norm, D is dense in E.

Now let's determine D^o. For $f \in D^o$, by definition, $f \in E$ and there exists $r > 0$ such that $BO(f, r) \subset D^o \subset D$. Let $g : x \mapsto f(x) + r \times x$. g belongs to $BO(f, r)$ because $\|g - f\|_{1,[0,1]} = \int_0^1 r \times x \, dx = \frac{r}{2} < r$. However, as $g(1) - g(0) = r \neq 0$, we obtain that $g \notin D^\circ$. Thus D has an empty interior, $D^\circ = \emptyset$.