Prove a function to be Riemann integrable

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Exercise 2.1.3. Consider the function $h : [0, 10] \to \mathbb{R}$ defined by $h(x) = 1$ if $x \in [\sqrt{2}, 2\sqrt{2}]$ and $h(x) = 0$ otherwise. By using regular partitions of $[0, 10]$, show that the function h is Riemann integrable on $[0, 10]$.

Solution

Let $n \in \mathbb{N}$ and define the regular partition $P_n = \{0, 10\frac{1}{n}, 10\frac{2}{n}, \dots, 10\frac{n-1}{n}, 10\}$. We are tring to prove $\sup_P L(f, P_n) = \inf_P U(f, P_n)$.

When n is big enough, it is evident that $\sqrt{2}$ and $2\sqrt{2}$ are not in the same interval $\left[\frac{10i}{n}, \frac{10(i+1)}{n}\right]$. We can find two values of i, named respectively i_1 and i_2 with $0 \le i_1+1 \le i_2 \le n-1$, so that $h(x) = 0$ for $x \in \left[0, \frac{10i_1}{n}\right]$, $\frac{10i_1}{n} \le \sqrt{2} \le \frac{10(i_1+1)}{n}$, $h(x) = 1$ for $x \in \left[\frac{10(i_1+1)}{n}, \frac{10i_2}{n}\right]$, $\frac{10i_2}{n} \le 2\sqrt{2} \le \frac{10(i_2+1)}{n}$ and $h(x) = 0$ for $x \in \left[\frac{10(i_2+1)}{n}, 10\right]$.

Based on the supremum and the infimum of the values on each interval $\left[\frac{10i}{n}, \frac{10(i+1)}{n}\right]$, we can calculate the lower and upper sums of f with respect to P_n:

$$L(f, P_n) = \sum_{i=0}^{i_1-1} 0\frac{10}{n} + \sum_{i=i_1}^{i_1} 0\frac{10}{n} + \sum_{i=i_1+1}^{i_2-1} 1\frac{10}{n} + \sum_{i=i_2}^{i_2} 0\frac{10}{n} + \sum_{i=i_2+1}^{n-1} 0\frac{10}{n} = (i_2 - i_1 - 1)\frac{10}{n} = \frac{10(i_2 - i_1 - 1)}{n}$$

A similar calculation gives

$$U(f, P_n) = \sum_{i=0}^{i_1-1} 0\frac{10}{n} + \sum_{i=i_1}^{i_1} 1\frac{10}{n} + \sum_{i=i_1+1}^{i_2-1} 1\frac{10}{n} + \sum_{i=i_2}^{i_2} 1\frac{10}{n} + \sum_{i=i_2+1}^{n-1} 0\frac{10}{n} = (i_2 - i_1 + 1)\frac{10}{n} = \frac{10(i_2 - i_1 + 1)}{n}$$

From $\frac{10i_1}{n} \le \sqrt{2} \le \frac{10(i_1+1)}{n}$ and $\frac{10i_2}{n} \le 2\sqrt{2} \le \frac{10(i_2+1)}{n}$, so $\frac{10(i_2-i_1-1)}{n} \le \sqrt{2} \le \frac{10(i_2-i_1+1)}{n}$

This indicates that

$$L(f, P_n) = \frac{10(i_2 - i_1 - 1)}{n} \ge \sqrt{2} - \frac{20}{n}$$

and

$$U(f, P_n) = \frac{10(i_2 - i_1 + 1)}{n} \le \sqrt{2} + \frac{20}{n}$$

We know that if P' is a finer partition of $[a, b]$, meaning that $P \subset P'$ (P' contains the points of

P and additional points, thus it contains more subdivisions of $[a, b]$), then one has

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

As $n \to \infty$, we have $L(f, P_n) \to \sqrt{2}$ and $U(f, P_n) \to \sqrt{2}$, so $\sup_{P_n} L(f, P_n) = \inf_{P_n} U(f, P_n) = \sqrt{2}$.

From this it follows that h is Riemann integrable on $[0, 10]$ and that the integral is $\sqrt{2}$.