The Interior and Closure for a Set of Matrices of a Certain Rank

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Here $K = \mathbb{R}$ or $\mathbb{C}$. We consider $M_n(K)$ equipped with a certain norm. We try to determine the interior and interior of:

$$A = \{M \in M_n(K), \text{rank}(M) = p\}, \text{ where } 0 \leq p \leq n.$$

The interior

Let $M \in A$. From the Gaussian pivot algorithm, we know that M is equivalent to a diagonal matrix D with p times the number 1 on the diagonal.

$$M = P \cdot \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \cdot Q$$

Let M_ϵ be defined by

$$M = P \cdot \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \epsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \epsilon \end{bmatrix} \cdot Q$$

Then $\|M - M_\epsilon\| \leq \|P\| \times \epsilon \times \|Q\|$ and $M_\epsilon \in GL_n(\mathbb{R})$. Thus, for all $M \in A$ and all $\delta > 0$, we have

$$BO(M, \delta) \cap A^c \neq \emptyset.$$

Hence $M \notin A^\circ$. Thus, if $p \neq n$, we have $A^\circ = \emptyset$.

If $p = n$, then $A = det^{-1}(\mathbb{R}^*)$, so A is open as the preimage of an open set by a continuous function.

The closure

Let M be a matrix of rank $k \leq p$. According to the Gauss elimination algorithm, we know that M is equivalent to a diagonal matrix D with k occurrences of the number 1 on the diagonal:

$$M = P \cdot \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \cdot Q$$

Let $\epsilon > 0$. We will add $(p-k)$ occurrences of the number ϵ on the diagonal. Let's define:

$$M_\epsilon = P \cdot \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \epsilon & \dots & 0 \\ 0 & 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \cdot Q$$

Then, M_ϵ is of rank p and:

$$\|M_\epsilon - M\| \leq \|P\| \cdot \|D_\epsilon - D\| \cdot \|Q\| \leq \epsilon \cdot \|P\| \cdot \|Q\|$$

Therefore,

$$\lim_{\epsilon \to 0} M_\epsilon = M$$

and thus $M \in \overline{A}$.

Consequently, we have:

$$\{M \in M_n(K), \text{rank}(M) \leq p\} \subset \overline{A}$$

Let us show that the set $\{M \in M_n(K), \text{rank}(M) \leq p\}$ is a closed set. To do this, we will demonstrate that its complement $\{M \in M_n(K), \text{rank}(M) > p\}$ is open.

Let M be a matrix of rank $k > p$. According to the Gauss elimination algorithm, we know that M is equivalent to a diagonal matrix D with k occurrences of the number 1 on the diagonal:

$$M = P \cdot \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \cdot Q$$

Let $\epsilon > 0$. We will consider the open ball $BO(M, \epsilon)$. Let $N \in BO(M, \epsilon)$, and let $D' = P^{-1} \cdot N \cdot Q^{-1}$ so that $N = P \cdot D' \cdot Q$. Then,

$$\|D - D'\| = \|P^{-1} \cdot (M - N) \cdot Q^{-1}\| \leq \|P^{-1}\| \cdot \epsilon \cdot \|Q^{-1}\|$$

We have:

$$D - D' = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

Furthermore,

$$\|D_1 - I_k\| \leq \|D - D'\| \leq \|P^{-1}\| \cdot \epsilon \cdot \|Q^{-1}\|$$

By the continuity of the determinant, for sufficiently small ϵ, we have $\det(D_1) > 0$. Thus, $\text{Rank}(D - D') \geq k$, and therefore,

$$BO(M, \epsilon) \subset \{M \in M_n(K), \text{rank}(M) > p\}$$

Thus, $\{M \in M_n(K), \text{rank}(M) > p\}$ is an open set, and consequently,

$$\{M \in M_n(K), \text{rank}(M) \leq p\} \text{ is closed.}$$

In conclusion, we have $\overline{A} = \{M \in M_n(K), \text{rank}(M) \leq p\}$. In particular, this proves that $\text{GL}_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$.