# The convergence of $L^p$ norm to $L^\infty$ norm for n-dimensional arrays and continuous functions defined in [0,1]

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## 1 For n-dimensional array

**Theorem**

In $\mathbb{R}^n$, we note: $\forall x \in \mathbb{R}^n, \|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{\frac{1}{p}}$ and $\|x\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$. Thus, $\forall x \in \mathbb{R}^n, \|x\|_p \xrightarrow[p\to+\infty]{} \|x\|_\infty$.

**Proof**

The result is evident for $x = 0$. Now we suppose $x \in \mathbb{R}^n$ a non-zero array. Then :

$$\forall p > 1, \|x\|_p = \|x\|_\infty \left( \left( \frac{x_1}{\|x\|_\infty} \right)^p + \left( \frac{x_2}{\|x\|_\infty} \right)^p + \ldots + \left( \frac{x_n}{\|x\|_\infty} \right)^p \right)^{\frac{1}{p}} \tag{1}$$

It is clear that there is a $k$ in $\{1, 2, 3, ..., n\}$ such that $|x_k| = \|x\|_\infty$ ($x_k$ is seen as the "dominant" one among $x_1, x_2, ..., x_n$). So we have:

$$1 \leq \left( \left( \frac{x_1}{\|x\|_\infty} \right)^p + \left( \frac{x_2}{\|x\|_\infty} \right)^p + \ldots + \left( \frac{x_n}{\|x\|_\infty} \right)^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{p}} \tag{2}$$

As $p \to \infty$, by using the squeeze theorem in the inequality, we obtain:

$$\left( \left( \frac{x_1}{\|x\|_\infty} \right)^p + \left( \frac{x_2}{\|x\|_\infty} \right)^p + \ldots + \left( \frac{x_n}{\|x\|_\infty} \right)^p \right)^{\frac{1}{p}} \xrightarrow[p\to+\infty]{} 1 \tag{3}$$

Using (1) and (3), we finally get:

$$\|x\|_p \xrightarrow[p\to+\infty]{} \|x\|_\infty \tag{4}$$

## 2 For continuous functions defined in [0,1]

**Theorem**

In $E = C^0([0,1], \mathbb{R})$, we note: $\forall f \in E, \|f\|_p = \left( \int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}}$ and $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$. Thus, $\forall f \in E, \lim_{p\to+\infty} \|f\|_p = \|f\|_\infty$.

**Proof**

The result is evident for $f = 0$. Now we assume that f is a function that is not constantly zero. We have:

$$\forall p > 1, \|f\|_p = \|f\|_\infty \left(\int_0^1 \left(\frac{|f(t)|}{\|f\|_\infty}\right)^p dt\right)^{\frac{1}{p}} \tag{5}$$

Suppose $\epsilon > 0$. A continuous function on a closed and bounded interval attains its bounds, so there exists $t_0 \in [0,1]$ such that $f(t_0) = \|f\|_\infty$.

Firstly we study the general case, which means $t_0 \in (0,1)$. As $|f|$ is continue at $t_0$, there thus exists $\eta > 0$ small enough such that $|t - t_0| < \eta \Rightarrow \Big||f(t)| - |f(t_0)|\Big| < \epsilon$.

So we have:

$$|t - t_0| < \eta \Rightarrow |f(t)| > |f(t_0)| - \epsilon \tag{6}$$

Then:

$$\left(\int_{t_0-\eta}^{t_0+\eta} (|f(t_0)| - \epsilon)^p \, dt\right)^{\frac{1}{p}} \leq \left(\int_0^1 (|f(t)|)^p \, dt\right)^{\frac{1}{p}} \tag{7}$$

Here, $\left(\int_{t_0-\eta}^{t_0+\eta} (|f(t_0)| - \epsilon)^p \, dt\right)^{\frac{1}{p}} = (2\eta)^{\frac{1}{p}}[(|f(t_0)| - \epsilon)^p]^{\frac{1}{p}} = (2\eta)^{\frac{1}{p}}(|f(t_0)| - \epsilon)$. By dividing two sides by $|f(t_0)|$, we have:

$$(2\eta)^{\frac{1}{p}}\left(1 - \frac{\epsilon}{\|f\|_\infty}\right) \leq \left(\int_0^1 \left(\frac{|f(t)|}{\|f\|_\infty}\right)^p dt\right)^{\frac{1}{p}} \leq 1 \tag{8}$$

When $t_0 = 0$ or $t_0 = 1$, things are being slightly different. We try to reach the same inequality (8). For $t_0 = 0$, we still set $\eta > 0$ small enough (at least less than $\frac{1}{2}$) so we have $\left(\int_{t_0}^{t_0+\eta} 2[(|f(t_0)| - \epsilon)^p]dt\right)^{\frac{1}{p}} \leq \left(\int_0^1 (|f(t)|)^p \, dt\right)^{\frac{1}{p}}$. Here, $\left(\int_{t_0}^{t_0+\eta} 2[(|f(t_0)| - \epsilon)^p]dt\right)^{\frac{1}{p}} = (2\eta)^{\frac{1}{p}}[(|f(t_0)| - \epsilon)^p]^{\frac{1}{p}} = (2\eta)^{\frac{1}{p}}(|f(t_0)| - \epsilon)$ and we get the inequality (8) again. The proof is similar for $t_0 = 1$. Therefore, the inequality (8) holds for all possible values of $t_0$.

It is clear that $(2\eta)^{\frac{1}{p}} \to 1$ when $p \to \infty$, so there exists $P \in \mathbb{N}$ such that $p \geq P \Rightarrow \left|(2\eta)^{\frac{1}{p}} - 1\right| < \epsilon$. Substituting this into the inequality (8), we have:

$$\forall p \geq P, (1 - \epsilon)\left(1 - \frac{\epsilon}{\|f\|_\infty}\right) \leq \left(\int_0^1 \left(\frac{|f(t)|}{\|f\|_\infty}\right)^p dt\right)^{\frac{1}{p}} \leq 1 \tag{9}$$

This shows that:

$$\forall p \geq P, 1 - \epsilon\left(1 + \frac{1}{\|f\|_\infty}\right) \leq \left(\int_0^1 \left(\frac{|f(t)|}{\|f\|_\infty}\right)^p dt\right)^{\frac{1}{p}} \leq 1 \tag{10}$$

Finally, we get:

$$\left(\int_0^1 \left(\frac{|f(t)|}{\|f\|_\infty}\right)^p dt\right)^{\frac{1}{p}} \xrightarrow[p \to +\infty]{} 1 \tag{11}$$

We conclude that:

$$\|f\|_p \xrightarrow[p \to +\infty]{} \|f\|_\infty \tag{12}$$