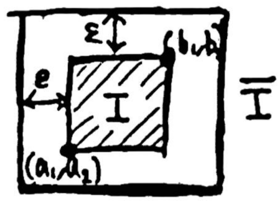
Outer Measure = Volume for n-box. Yat Ming Luk

$I = \prod_{i=1}^{n} [a_i, b_i]$, $\mathrm{vol}(I) = \prod_{i=1}^{n} (b_i - a_i)$

$m^*(V) = \inf \{\sigma(S) \mid S \in C_V\}$, C_V = all coverings of V.



<u>Claim:</u> $m^*(I) = \mathrm{vol}(I)$

<u>Proof ($m^*(I) \leq \mathrm{vol}(I)$):</u>

Let us consider a box $\bar{I} = \prod_{i=1}^{n} [a_i - \varepsilon, b_i + \varepsilon)$ where a_i, b_i are the same as in I, and $\varepsilon > 0$. Clearly. $I \subset \bar{I}$, so $\bar{I}$ is a covering of I. $\bar{I}$ is also technically a "finite union of boxes", so $\sigma(\bar{I}) = \mathrm{vol}(\bar{I})$.

Then $m^*(I) = \inf\{\sigma(S) \mid S \text{ covering } I\} \leq \sigma(\bar{I}) = \mathrm{vol}(\bar{I}) =$

$= \prod_{i=1}^{n} (b_i - a_i + 2\varepsilon)$. But ε is arbitrary, so we have that $m^*(I) \leq \prod_{i=1}^{n} (b_i - a_i) = \mathrm{vol}(I)$ as required.

<u>Proof ($\mathrm{vol}(I) \leq m^*(I)$):</u>

Let us consider $\{I_j\}_j$, a countable covering of I, then for each j, let $\bar{I}_j$ be a cube with the properties.

- $(1 + \varepsilon)\mathrm{vol}(I_j) \geq \mathrm{vol}(\bar{I}_j)$, $\varepsilon > 0$ (*)
- $I_j \subseteq \mathrm{int}\, \bar{I}_j$ (int means interior of.)

Since the interior of a set is always open, and $\bigcup \mathrm{int}\, \bar{I}_j$ is a cover of I, $\bigcup \mathrm{int}\, \bar{I}_j$ is an open cover of I, and I is clearly compact, since $[a_i, b_i] \subseteq \mathbb{R}$.

Therefore, $\bigcup \operatorname{int} \bar{I}_j$ must have a finite subcover. $\Rightarrow$

$$I \subseteq \bigcup_{k=1}^{M} \operatorname{int} \bar{I}_j \subseteq \bigcup_{j=1}^{M} \bar{I}_j \quad \text{for some } M>0.$$

Recalling that if $A \subseteq \bigcup_j I_j$, then $\operatorname{vol}(A) \leq \sum_j \operatorname{vol}(I_j)$, we have

$$\operatorname{vol}(I) \leq \sum_{j=1}^{M} \operatorname{vol}(\bar{I}_j) \overset{*}{\leq} (1+\varepsilon)\sum_{j=1}^{M} \operatorname{vol}(I_j) \leq (1+\varepsilon) \sum_j \operatorname{vol}(I_j)$$

However, $\{I_j\}_j$ is arbitrary, so we choose a covering such that $\sum_j \operatorname{vol}(I_j) = m^*(I) + \varepsilon$. $\Rightarrow$ $\operatorname{vol}(I) \leq (1+\varepsilon)(m^*(I)+\varepsilon)$

But ε is arbitrary, so we have $\operatorname{vol}(I) \leq m^*(I)$ as required.

$$\left(m^*(I) \leq \operatorname{vol}(I) \wedge m^*(I) \geq \operatorname{vol}(I)\right) \longrightarrow m^*(I) = \operatorname{vol}(I) \quad \blacksquare$$

Sources:

- Heil, C. (2019). Introduction to Real Analysis. Springer.