LUK Yat Ming
062101906

Proof that δ_y, T_h, and δ_y^α are distributions.

Remark: A sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to $f_\infty \in \mathcal{D}(\mathbb{R}^n)$ if:

1) $\sup_{x\in\mathbb{R}^n} |\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)| \xrightarrow{j\to\infty} 0 \quad \forall \alpha \in \mathbb{N}$
2) $\mathrm{supp}(f_j) \subset B(0,R)$ for some $R\in\mathbb{R}$ and $\forall j \in \mathbb{N}$

Claim 1: δ_y^α is a distribution, $(\delta_y^\alpha[f] := \partial^\alpha f(y))$ for any $\alpha \in \mathbb{N}$.

Proof "A": Let us show δ_y^α is linear. Let $f_1, f_2 \in \mathcal{D}(\mathbb{R}^n)$, $\lambda \in \mathbb{F}$,

then $\delta_y^\alpha[f_1 + \lambda f_2] = [\partial^\alpha f_1 + \partial^\alpha \lambda f_2](y) = \partial^\alpha f_1(y) + \lambda \partial^\alpha f_2(y) =$

$= \delta_y^\alpha[f_1] + \lambda \delta_y^\alpha[f_2]$ $\boxed{A}$

Proof "B": Let us show that $\delta_y^\alpha[f_j] \xrightarrow{j\to\infty} \delta_y^\alpha[f_\infty]$ if $f_j \xrightarrow{j\to\infty} f_\infty$.

We can equivalently show that $|\delta_y^\alpha[f_j] - \delta_y^\alpha[f_\infty]| \xrightarrow{j\to\infty} 0$

$$|\delta_y^\alpha[f_j] - \delta_y^\alpha[f_\infty]| = |\partial^\alpha f_j(y) - \partial^\alpha f_\infty(y)| \le \sup_{x\in\mathbb{R}^n} |\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)| \xrightarrow[j\to\infty]{(1)} 0$$

since we know $\{f_j\}_{j\in\mathbb{N}}$ converges. $\boxed{B}$. A+B is sufficiently sufficient. $\blacksquare$

Claim 2: δ_y is a distribution $(\delta_y[f] := f(y))$.

This is a special case of δ_y^α with $\alpha = 0$. Therefore, we have shown the claim as required. $\blacksquare$

Claim 3: T_h is a distribution, where $T_h(f) := \int_{\mathbb{R}^n} f(x)h(x)\,dx$ and $h \in L^1_{loc}(\mathbb{R}^n)$

Proof "A": Let us show T_h is linear. Let $f_1, f_2 \in \mathcal{D}(\mathbb{R}^n)$, $\lambda \in \mathbb{F}$, then $T_h[f_1 + \lambda f_2] = \int_{\mathbb{R}^n} [f_1 + \lambda f_2](x)\,h(x)\,dx = \int_{\mathbb{R}^n} f_1(x)h(x)\,dx + \lambda \int_{\mathbb{R}^n} f_2(x)h(x)\,dx$

$= T_h[f_1] + \lambda T_h[f_2]$ A

Proof "B": Let us show $T_h[f_j] \xrightarrow{j\to\infty} T_h[f_\infty]$ if $f_j \xrightarrow{j\to\infty} f_\infty$

Equivalently we show $|T_h[f_j] - T_h[f_\infty]| \xrightarrow{j\to\infty} 0$

$|T_h[f_j] - T_h[f_\infty]| = \left| \int_{\mathbb{R}^n} f_j(x)h(x)\,dx - \int_{\mathbb{R}^n} f_\infty(x)h(x)\,dx \right|$

$= \left| \int_{\mathbb{R}^n} h(x)(f_j(x) - f_\infty(x))\,dx \right| \leq \int_{\mathbb{R}^n} |h(x)||f_j(x) - f_\infty(x)|\,dx$

$\leq \sup_{x' \in \mathbb{R}^n} |f_j(x') - f_\infty(x')| \int_{B(0,R)} |h(x)|\,dx = \sup_{x' \in \mathbb{R}^n} |f_j(x') - f_\infty(x')| \int_{B(0,R)} |h(x)|\,dx$

$\underset{(2)}{\overset{(1)}{\xrightarrow{j\to\infty}}} 0$ since we know $\operatorname{supp}(f_j) \subset B(0,R)$ and since $\int_{B(0,R)} |h(x)|\,dx < \infty$

$\Rightarrow T_h[f_j] \xrightarrow{j\to\infty} T_h[f_\infty]$ B. A+B is sufficient