On Lebesgue Measures

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This document will discuss an alternative approach on defining Lebesgue measures by defining and proving some properties of the Lebesgue inner measure, and prove some properties of the Lebesgue measure by using an approach not covered in the lecture notes.

1 Lebesgue Outer Measure

1.1 Definition

First, the definition of the Lebesgue Outer Measure (as written in the lecture notes):

Definition of Lebesgue Outer Measure (**Definition 2.2.3.**)

For a subset $\Omega \subset \mathbb{R}^n$, the **Lebesgue outer measure** of Ω is defined by

$$m^*(\Omega) := \inf \{\sigma(S) \mid S \text{ covering of } \Omega\}.$$

Equivalently, one has for any $\varepsilon > 0$, there exists a covering S of Ω such that

$$m^*(\Omega) \leq \sigma(S) \leq m^*(\Omega) + \varepsilon.$$

In the lecture, we considered only coverings of $\Omega \in \mathbb{R}^n$ with *closed* boxes (or interval). However, observe that the volume of a box I is

$$v(I) = \prod_{j=1}^{n}(b_j - a_j)$$

regardless if the box is open, closed, or half-closed, one can always consider the covering of Ω with open sets.

This statement will be proven for $\Omega \subset \mathbb{R}$, but this proof can be generalized to any $\Omega \subset \mathbb{R}^n$. One has:

$$m^*(\Omega) = \inf \Big\{ \sum_j v(I_j) \mid \Omega \subset \bigcup_j I_j \Big\}.$$

Here, the I_j are assumed to be closed intervals of the form $[a_j, b_j]$, and as such, for any $\varepsilon > 0$, there exists a covering of Ω, $S = \cup_j I_j$ such that

$$m^*(\Omega) \leq \sigma(S) = \sum_j (b_j - a_j) \leq m^*(\Omega) + \varepsilon.$$

For every I_j, one then sets I'_j to be the open interval with

$$I'_j = \left(a_j - \tfrac{\varepsilon}{2^{j+2}}\ ,\ b_j + \tfrac{\varepsilon}{2^{j+2}}\right) \equiv \{x \in \mathbb{R} \mid a_j - \tfrac{\varepsilon}{2^{j+2}} < x < b_j + \tfrac{\varepsilon}{2^{j+2}}\}.$$

The length of all I'_j is:

$$v(I'_j) = \left(b_j + \frac{\varepsilon}{2^{j+2}}\right) - \left(a_j - \frac{\varepsilon}{2^{j+2}}\right) = (b_j - a_j) + \frac{\varepsilon}{2^{j+1}}.$$

One sets S' to be the union of all such I'_j. Since $I_j \subset I'_j$, S' is also a covering of Ω. As such,

$$\sigma(S') = \sum_j v(I'_j) = \sum_j \left((b_j - a_j) + \frac{\varepsilon}{2^{j+1}}\right) \leq \frac{\varepsilon}{2} + \sum_j (b_j - a_j) = \sigma(S) + \frac{\varepsilon}{2}.$$

The last inequality is the result of:

$$\sum_j \frac{\varepsilon}{2^j} \leq \varepsilon \sum_j^{\infty} \frac{1}{2^j} = \varepsilon. \qquad (♥)$$

Therefore, for all $\varepsilon > 0$, one has that:

$$m^*(\Omega) \leq \sigma(S) \leq \sigma(S') \leq m^*(\Omega) + \varepsilon.$$

As such, one can consider either coverings consisting of open or closed intervals without any problems. By generalizing this proof to $\mathbb{R}^n$, one can consider coverings consisting of either open or closed boxes.

1.2 Properties of the Lebesgue Outer Measure

Properties of the Lebesgue Outer Measure (Exercise 2.2.5)

1. If $\Omega_1 \subset \Omega_2$, then $m^*(\Omega_1) \leq m^*(\Omega_2)$. **(Monotonicity)**
2. $m^*(\cup_j \Omega_j) \leq \sum_j m^*(\Omega_j)$ for a finite or countable family. **(Subadditivity)**
3. For a pairwise disjoint family of (closed and) compact sets $(K_j)_j$, $m^*(\cup_j K_j) = \sum_j m^*(K_j)$.

1.2.1 Proof of Monotonicity of the Lebesgue Outer Measure

Observe that since $\Omega_1 \subset \Omega_2$, any covering S of Ω_2 satisfies:

$$\Omega_1 \subset \Omega_2 \subset S,$$

and as such, for any S,

$$m^*(\Omega_1) \leq \sigma(S).$$

By the definition of the outer measure of Ω_2,

$$m^*(\Omega_1) \leq \inf\{\sigma(S) \mid S \text{ covering of } \Omega_2\} = m^*(\Omega_2). \quad \square$$

1.2.2 Proof of Subadditivity of the Lebesgue Outer Measure

Before proving subadditivity, notice that for any two boxes I_1 and I_2,

$$v(I_1 \cup I_2) = v(I_1) + v(I_2) - v(I_1 \cap I_2) \leq v(I_1) + v(I_2).$$

Since coverings are made up of such boxes, one has that for any family of coverings $\{S_j\}_j$,

$$\sigma(\cup_j S_j) \leq \sum_j \sigma(S_j).$$

By the definition of the outer measure, for every Ω_j and any $\varepsilon > 0$, there exists a covering S_j of Ω_j such that

$$m^*(\Omega_j) \leq \sigma(S_j) \leq m^*(\Omega_j) + \frac{\varepsilon}{2^{j+1}}.$$

Since $\Omega_j \subset S_j$ for all j, then $\cup_j \Omega_j \subset \cup_j S_j$ (i.e., $\cup_j S_j$ is a covering of $\cup_j \Omega_j$), and as such,

$$m^*(\cup_j \Omega_j) \leq \sigma(\cup_j S_j) \leq \sum_j \sigma(S_j) \leq \sum_j \left(m^*(\Omega_j) + \frac{\varepsilon}{2^{j+1}}\right) \leq \frac{\varepsilon}{2} + \sum_j m^*(\Omega_j).$$

The last inequality is a result of (♥), and this proves the statement (as ε can be taken arbitrarily small).

1.2.3 Lebesgue Outer Measure of a Family of Pairwise Disjoint Compact Sets

Being pairwise-disjoint means that for any $i \neq k$ in the range of numbers j can be,

$$\Omega_i \cap \Omega_k = \emptyset.$$

To construct the proof of the statement, we need some way of expressing the distance between two sets.

One way that comes to mind is a distance function. For two points $x, y \in \mathbb{R}^n$,

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2},$$

and the distance between two sets Ω_1 and Ω_2 is defined as

$$d(\Omega_1, \Omega_2) = \inf \{ d(x,y) \mid x \in \Omega_1, y \in \Omega_2 \}.$$

For two disjoint compact sets K_1 and K_2, $d(K_1, K_2) > 0$. While this might seem intuitive for any set, in fact, it is not always true for all sets, but is always true for compact sets. For example, consider $\Omega_1 = \{(x,y) \in \mathbb{R}^2 \mid y \leq 0\}$ and $\Omega_2 = \{(x,y) \in \mathbb{R}^2 \mid y = |x|^{-1}, x \neq 0\}$. While these two sets are disjoint, $d(\Omega_1, \Omega_2) = 0$.

Disjoint Compact Sets are Separated by a Positive Distance

A compact set Ω in $\mathbb{R}^n$ is a set where any sequence taking values in Ω will always converge to a point within Ω. Consider two disjoint compact sets K_1 and K_2. By definition of disjoint sets,

$$K_1 \cap K_2 = \emptyset.$$

Suppose that $d(K_1, K_2) = 0$. Then, consider the continuous function

$$d(x, K_2) = \inf \{ d(x,y) \mid y \in K_2 \}.$$

If $d(K_1, K_2) = 0$, then, there exists $x_1 \in K_1$ such that $d(x_1, K_2) = 0$. As such,

$$K_1 \ni x \in \mathrm{cl}(K_2) = K_2 \Longrightarrow K_1 \cap K_2 \neq \emptyset.$$

This presents us with a contradiction of the assumption that K_1 and K_2 are disjoint. □

The statement that we want to prove is that if $\{K_j\}_j$ are pairwise disjoint, compact sets, then

$$m^*(\cup_j K_j) = \sum_j m^*(K_j).$$

From the subadditivity of the outer measure, we have that

$$m^*(\cup_j K_j) \leq \sum_j m^*(K_j),$$

and as such, we only need to prove the reverse inequality.

For any arbitrary $\varepsilon > 0$, there exists a covering S of $\cup_j K_j$ such that

$$\sigma(S) < m^*(\cup_j K_j) + \varepsilon.$$

Since $d(K_i, K_k) > 0$ for all $i \neq k$ on the range of numbers j can take, we can construct a new covering of $\cup_j K_j$ by intervals of length less than $\frac{1}{2} \inf d(K_i, K_k)$ only, and call this new covering S^*. Of course, by breaking up the intervals, we do not change the value of σ, so

$$\sigma(S) = \sigma(S^*).$$

Now, we construct a covering of $\cup_j K_j$ based on the following method:

$$S_0 = \{I \in S^* \mid I \cap (\cup_j K_j) = \emptyset\}$$
$$S_i = \{I \in S^* \mid I \cap (\cup_j K_j \backslash K_i) = \emptyset\}$$

This method separates the covering S^* based on which K_j they are intersecting, with S_0 being the part of S^* that do not touch any of the K_j, and S_i being the part of S^* that touches their corresponding K_i without touching any other K_j. Since the boxes that construct S^* are taken to be small and of side (edge) length less than half of $\inf d(K_i, K_k)$, this construction of S_0 and the S_i is always possible.

By constructing the S_i for all j, all S_i are coverings of their corresponding K_i by closed boxes, and as such,

$$\sum_j m^*(K_j) \leq \sum_j \sigma(S_j) \leq \sigma(S_0) + \sum_j \sigma(S_j) = \sigma(S^*) < m^*(\cup_j K_j) + \varepsilon.$$

As ε is arbitrary (and can be taken to be very small),

$$\sum_j m^*(K_j) \leq m^*(\cup_j K_j).$$

Combining this with the previous inequality, we conclude that:

$$m^*(\cup_j K_j) = \sum_j m^*(K_j).$$

1.3 Outer Measure of a Finite Box

Outer Measure of a Box (Exercise 2.2.4.)

For any box $I \subset \mathbb{R}^n$, one has that:

$$m^*(I) = v(I).$$

While this might seem obvious and trivial, the proof of this is not trivial at all.

Since I is a closed box, I is of the form

$$I = \big\{x \in \mathbb{R}^n \mid c_j \leq x \leq d_j \text{ for } j \in \{1, \ldots, N\}\big\}.$$

For convenience, a new notation for boxes in $\mathbb{R}^n$ will be introduced. Set:

$$\text{box}(a, b) := \{x \in \mathbb{R}^n \mid a_j < x_j < b_j \text{ for } j \in \{1, \ldots, n\}\}$$

for open boxes and

$$\text{box}[a, b] := \{x \in \mathbb{R}^n \mid a_j \leq x_j \leq b_j \text{ for } j \in \{1, \ldots, n\}\}$$

for closed boxes. First, one observes that I can be covered by I itself, so one has:

$$m^*(I) \leq v(I). \tag{1}$$

To prove the desired equality, one needs only to show that:

$$v(I) \leq m^*(I),$$

or equivalently, for any covering S of I consisting of open boxes $\{I_k\}_k$,

$$v(I) \leq \sigma(S) = \sum_k v(I_k).$$

Observe that here, the collection of open boxes $\{I_k\}_k$ can be either finite or countable. This complicates things a bit, since what works for finite collections might not work for countable sets. Fortunately, one can consider only covers made up of a finite number of open boxes by using the **Heine-Borel Theorem**:

Heine-Borel Theorem

Any closed and bounded subset $\Omega \subset \mathbb{R}^n$ is compact, and every open cover of Ω admits a finite subcover.

Clearly, I is closed and bounded, so one can reduce the complexity of this problem to only proving:

$$v(I) \leq \sigma(S) = \sum_{k=1}^{m} v(I_k). \tag{2}$$

For any cover S of a box I consisting of finitely many other boxes $\{I_k\}_k$, we can always create a new cover of the box I (denoted by S^*) consisting of a finite number of almost-disjoint boxes $\{\tilde{I}_j\}_j$ such that for all j, $\tilde{I}_j \subseteq I$. Almost-disjoint means that the boxes do not overlap each other except on their boundaries.

The method to achieve this is illustrated below for the case of boxes in $\mathbb{R}^2$ (but it works fine for $\mathbb{R}^n$):

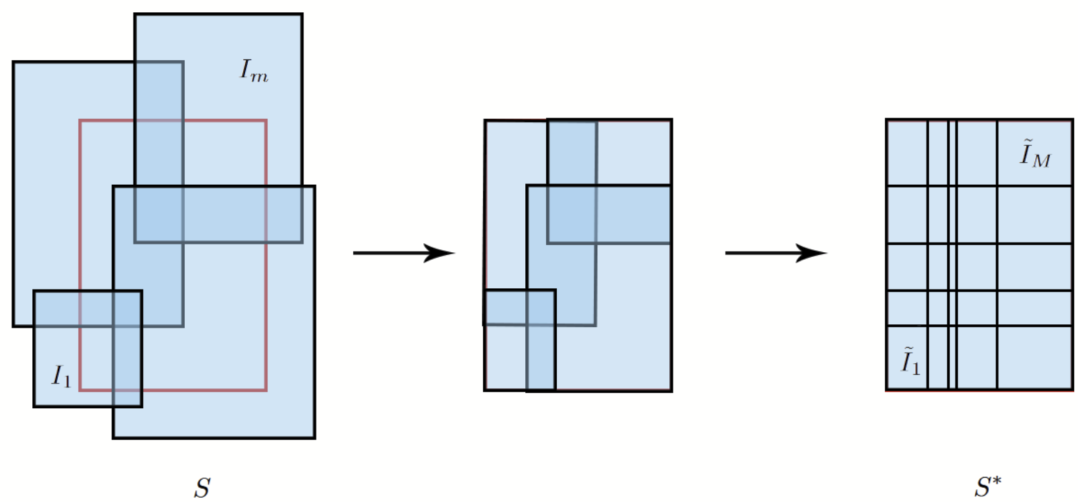

Figure 1: Creating S^* from an arbitrary S. I is the red rectangle in the left picture.

The method of creating S^* from an arbitrary S is as follows:

1. Create an (intermediate) cover of I by:
$$\bigcup_{k=1}^{m} I_m \cap I.$$
This cover will also be a cover by a finite number of boxes. Note that in this step, overlap between multiple boxes is still present, but the volume of this cover will be less or equal to the original cover.

2. From the cover obtained in Step 1, extend indefinitely the edges of all of the boxes in that cover. By doing this, we obtain a n-dimensional grid that will separate the box I into finitely-many (say, M) boxes $\{\tilde{I}_j\}_j$. The cover of I by these boxes is exactly S^*.

By construction, we have that:

$$I = \bigcup_{j=1}^{M} \{\tilde{I}_j\}_j$$

and that

$$\sigma(S^*) \leq \sigma(S),$$

because when constructing S^*, we removed all parts where two or more boxes overlap, and this means that the total volume of the $\{\tilde{I}_j\}_j$ (i.e. $\sigma(S^*)$ is less than the total volume of the $\{I_k\}_k$.

Notice that the grid partitions the sides of I, and each $\tilde{I}_j$ consists of taking products of the (length of the) sides in these partitions. Therefore, the sum of the volumes of the $\tilde{I}_j$ is exactly the volume of I. Therefore,

$$\sigma(S^*) = \sum_{j=1}^{M} v(\tilde{I}_j) = v(I) \leq \sigma(S).$$

Since S is an arbitrary covering of I, then we have that:

$$v(I) \leq m^*(I).$$

Combining this with Equation (1), we obtained the desired result:

$$v(I) = m^*(I). \quad \square$$

When talking about the volume of an open box, we need only to consider its closure. Since the closure (of an open box) does not change the volume of the box, it also does not change its outer measure (as long as the box is finite). As such, $m^*(I) = v(I)$ is true for any box.

2 Lebesgue Inner Measure

2.1 Definition

In contrast to the Lebesgue outer measure, which seeks to approximate the size of Ω from the outside using a union of open sets and taking the infimum of such approximations, the Lebesgue inner measure seeks to approximate the size of Ω from the inside using closed sets. It is defined as:

Definition of Lebesgue Inner Measure

For a subset $\Omega \subset \mathbb{R}^n$, the **Lebesgue inner measure** of Ω is defined by

$$m_*(\Omega) := \sup \{m^*(K) \mid K \subset \Omega,\ K \text{ compact}\}.$$

By the Monotonicity of the Lebesgue Outer Measure (Section 1.2.1), one has:

$$m_*(\Omega) \leq m^*(\Omega),$$

since $m^*(K) \leq m^*(\Omega)$ as $K \subset \Omega$.

2.2 Inner Measure of a Box

Inner Measure of a Box

For any box $I \subset \mathbb{R}^n$, one has that:

$$m_*(I) = v(I).$$

Once again, let $I = \text{box}(c, d) \subset \mathbb{R}^n$ be a finite box (does not matter if it is open or closed). If $K \subset I$ is compact, then I is a cover of K, and by the definition of the Lebesgue outer measure,

$$m^*(K) \leq v(I),$$

and by the definition of the Lebesgue inner measure,

$$m_*(I) \leq v(I). \tag{3}$$

For any $\varepsilon > 0$, $\varepsilon < 1$ small enough to satisfy

$$\varepsilon < \min_{j \in \{1,\ldots,n\}} \frac{1}{2}(d_j - c_j),$$

the closed box I_ε defined by

$$I_\varepsilon := \{x \in \mathbb{R}^n \mid c_j + \varepsilon \leq x_j \leq d_j - \varepsilon\}$$

is compact, and satisfies:

$$m^*(I_\varepsilon) = v(I) - O(\varepsilon^k) \leq m_*(I),$$

where $O(\varepsilon^k)$ is a polynomial of degree n in ε (crucially, this polynomial consists of terms from ε^1 to ε^n, and no constant terms). Taking the limit as $\varepsilon \to 0$, we have that

$$v(I) \leq m_*(I). \tag{4}$$

Combining (3) and (4), we obtain the result:

$$m_*(I) = v(I).$$

3 Lebesgue-Measurable Sets and Lebesgue Measures

3.1 Definition

Definition of Lebesgue-Measurable Sets and Lebesgue Measure

A set $\Omega \subset \mathbb{R}^n$ is **Lebesgue-measurable** if

$$m_*(\Omega) = m^*(\Omega).$$

If Ω is Lebesgue-measurable, we set its Lebesgue measure $m(\Omega)$ to be:

$$m(\Omega) = m_*(\Omega) = m^*(\Omega).$$

If $m^*(\Omega) = \infty$, then Ω is Lebesgue-measurable if all $\Omega \cap \text{box}[-a, a]$ with $a_j > 0$ for all $j \in \{1, \ldots, N\}$ are Lebesgue-measurable.

One consequence is that all compact sets K are Lebesgue-measurable, since all compact sets are contained in itself, $m_*(K) = m^*(K)$. Furthermore, we have proven that the outer and inner measures of a box satisfy

$$m^*(I) = v(I) = m_*(I).$$

Therefore, all boxes (regardless if they are open or closed) are Lebesgue-measurable.

3.2 Lebesgue Measure of Pairwise-Disjoint Family of Sets

Additivity of Lebesgue Measures of Pairwise-Disjoint Family of Sets (Theorem 2.2.9.)

If $\Omega = \cup_j \Omega_j$ is a finite (or countable) union of pairwise-disjoint Lebesgue-measurable sets Ω_i, then Ω is Lebesgue-measurable, and its measure satisfies

$$m(\Omega) = \sum_j m(\Omega_j).$$

Here, we assume that $m(\Omega) < \infty$. When $m(\Omega) = \infty$, then this statement will hold for all $\Omega \cap \text{box}[-a, a]$ and $\Omega_j \cap \text{box}[-a, a]$ for all $a \in \mathbb{R}^n$ with $a_k > 0$ for all $k \in \{1, \ldots, n\}$.

First, consider the finite union case, and assume $j \in \{1, \ldots, N\}$ (i.e., there are N sets in the union).

By the subadditivity of the outer measure (Subsection 1.2.2), we have that:

$$m^*(\Omega) \leq \sum_j^N m^*(\Omega_j) = \sum_j^N m(\Omega_j).$$

Since all Ω_j are Lebesgue-measurable, for any $\varepsilon > 0$, we can choose $K_j \subset \Omega_j$ such that

$$m^*(K_j) \geq m_*(\Omega_j) - \frac{\varepsilon}{2^j}.$$

Since the $\{\Omega_j\}_j$ are pairwise disjoint, then by construction, the $\{K_j\}_j$ are also pairwise disjoint. As such,

$$m^*(\cup_j K_j) = \sum_j^N m^*(K_j).$$

The finite union of compact sets are compact. Therefore, $\cup_j K_j$ is compact, and $\cup_j K_j \subset \cup_j \Omega_j = \Omega$. Therefore,

$$m_*(\Omega) \geq m^*(\cup_j K_j) = \sum_j^N m^*(K_j) \geq \sum_j^N \left(m(\Omega_j) - \frac{\varepsilon}{2^j} \right) \geq \left(\sum_j m(\Omega j) \right) - \varepsilon.$$

The last inequality is a result of (♥) again. The countable union case is obtained by letting $N \to \infty$. Furthermore, since this equality holds for any $\varepsilon > 0$, we observe:

$$m_*(\Omega) \geq \sum_j m(\Omega_j) \geq m^*(\Omega).$$

However, previously, we have proven that $m_*(\Omega) \leq m^*(\Omega)$ for any set. As such, we conclude that:

$$m_*(\Omega) = m^*(\Omega),$$

and therefore, Ω is a Lebesgue-measurable set with $m(\Omega) = \sum_j m(\Omega_j)$. □

When $m(\Omega) = \infty$, replace Ω in this proof with $\Omega \cap I$ and Ω_j with $\Omega_j \cap I$ for all box$[-a, a]$ with $a_k > 0\ \forall k$.

3.3 All Open and Closed Sets are Lebesgue Measurable

In this section, we will prove that all open sets and all closed sets are Lebesgue-measurable. The proofs will consider $\Omega \subset \mathbb{R}$ for the ease of notation and to reduce clutter, but the proofs can be extended to $\Omega \subset \mathbb{R}^n$ without any issue (except, cluttering and/or making a mess).

3.3.1 All Open Sets are Lebesgue-Measurable

We will prove that all open sets $\Omega \subset \mathbb{R}$ can be written as a countable union of disjoint open intervals I_j.

Assume that $\Omega \neq \emptyset$ (in the case that $\Omega = \emptyset$, then it is already an open interval, $\emptyset$, and is Lebesgue-measurable and of measure 0 since any interval, no matter how arbitrarily small it is, will contain Ω).

For $x_1, x_2 \in \mathbb{R}$, define an equivalence relation $x_1 \sim x_2$ if and only if

$$[\min(x_1, x_2), \max(x_1, x_2)] \subseteq \Omega.$$

Before moving on, a brief reminder on equivalence relations and equivalence classes:

Equivalence Relations and Equivalence Classes

A relation $\sim$ is an equivalence relation on a set A if and only if, for all $a, b, c \in A$:
a) $a \sim a$ (symmetry)
b) If $a \sim b$ then $b \sim a$ (reflexivity)
c) If $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity)
An equivalence class (e.g., of $a \in A$) is defined as $[a] := \{b \in A \mid b \sim a\}$.

Let us now prove that this relation indeed defines an equivalence relation by checking the three properties:

a) If $x_1 \in \Omega$, then $[\min(x_1, x_1), \max(x_1, x_1)] = [x_1, x_1] = \{x_1\} \subseteq \Omega$ (i.e., $x_1 \sim x_1$).

b) $x_1 \sim x_2$ implies that $[\min(x_1, x_2), \max(x_1, x_2)] = [\min(x_2, x_1), \max(x_2, x_1)] \subseteq \Omega$ (i.e., $x_2 \sim x_1$).

c) $x_1 \sim x_2$ and $x_2 \sim x_3$ implies that $[\min(x_1, x_2), \max(x_1, x_2)] \subseteq \Omega$ and $[\min(x_2, x_3), \max(x_2, x_3)] \subseteq \Omega$. As both are subsets of Ω, their union is also a subset of Ω. Their union is given by:

$$[\min(x_1, x_2), \max(x_1, x_2)] \cup [\min(x_2, x_3), \max(x_2, x_3)] = [\min(x_1, x_2, x_3), \max(x_1, x_2, x_3)] \subseteq \Omega.$$

Furthermore, we have that:

$$\Omega \supseteq [\min(x_1, x_2, x_3), \max(x_1, x_2, x_3)] \supseteq [\min(x_1, x_3), \max(x_1, x_3)] \Longrightarrow x_1 \sim x_3.$$

As such, $\sim$ indeed does define an equivalence relation, with the equivalence classes being made up of open, disjoint intervals (since Ω is an open set). Set $\mathcal{P}$ to be the set of all equivalence classes, with $\Omega = \cup_{I \in \mathcal{P}} I$.

Now, we only need to prove that $\mathcal{P}$ is a countable set. For each $I \in \mathcal{P}$, choose an arbitrary $q_I \in \mathbb{Q}$ in I (we can always find two real numbers in an open interval, and we can always find a rational number between any two real numbers). We then observe that the map $\mathcal{P} \to \mathbb{Q} : I \to q_I$ is injective (since the I comprising $\mathcal{P}$ is pairwise-disjoint, so any q_I cannot belong to two or more intervals in $\mathcal{P}$). Since $\mathbb{Q}$ (the codomain) is countable, then $\mathcal{P}$ is countable. All intervals are measurable, so by **Theorem 2.2.9.**, Ω is Lebesgue-measurable since it is a union of pairwise-disjoint open intervals.

To extend this argument to $\Omega \subset \mathbb{R}^n$, we change 'intervals' to 'boxes', and change the definition of the equivalence relation $a \sim b$ for $a, b \in \mathbb{R}^n$ by:

$$a \sim b \iff \text{box}[c, d] \subseteq \Omega \quad \text{where } c_j = \min(a_j, b_j) \text{ and } d_j = \max(a_j, b_j) \text{ for all } j \in 1, \dots, N.$$

3.3.2 All Closed Sets are Lebesgue-Measurable

If $\Omega \subset \mathbb{R}$ is closed with $m^*(\Omega) = \infty$, then all $\Omega \cap [-a, a]$ is compact (since it is the intersection of a closed set Ω and a compact set $[-a, a]$). As all compact sets are Lebesgue-measurable, then Ω is also Lebesgue-measurable.

Therefore, let us restrict our discussion henceforth to Ω closed with $m^*(\Omega) < \infty$.

For any (arbitrary) given $\varepsilon > 0$ construct, for all $n \in \mathbb{N}$:

$$J_{n,\varepsilon} := \left(\Omega \cap \left[-n + \frac{\varepsilon}{2^{n+1}}, -n+1\right]\right) \cup \left(\Omega \cap \left[n-1, n - \frac{\varepsilon}{2^{n+1}}\right]\right).$$

Then, set the union of all such $J_{n,\varepsilon}$ to be

$$J_\varepsilon := \bigcup_{n \in \mathbb{N}} J_{n,\varepsilon}.$$

Since Ω is closed, its intersection with an interval is compact, and as union of compact sets are compact, all $J_{n,\varepsilon}$ are compact. Therefore, all $J_{n,\varepsilon}$ are Lebesgue-measurable, and since they are disjoint by construction, J_ε is a union of disjoint Lebesgue-measurable sets, and thus is Lebesgue-measurable (Section 3.2).

Observe that in the construction of J_ε, the gaps between the intervals have decreasing length, and by (♥), the total length of the gaps are exactly ε. Therefore,

$$m(J_\varepsilon) + \varepsilon = m^*(J_\varepsilon) + \varepsilon \geq m^*(\Omega) \geq m^*(J_\varepsilon) = m(J_\varepsilon) = \sum_n m(J_{n,\varepsilon}).$$

Since $m^*(\Omega) < \infty$, the countably infinite sum (over $n \in \mathbb{N}$) converges, and by considering only a finite number of terms, then the (finite) sum will have a value of at least $m(J_\varepsilon) - \varepsilon$. The corresponding union of $J_{n,\varepsilon}$ will be a compact set K_ε contained in Ω with

$$m(K_\varepsilon) \geq m^*(\Omega) - 2\varepsilon.$$

Since K_ε is a compact subset of Ω, we have that:

$$m(K_\varepsilon) \leq m_*(\Omega) \implies m_*(\Omega) \geq m^*(\Omega) - 2\varepsilon.$$

Since ε is arbitrary, then we conclude that

$$m_*(\Omega) \geq m^*(\Omega),$$

and combining this with the previously-proven result that $m_*(\Omega) \leq m^*(\Omega)$, we conclude:

$$m^*(\Omega) = m_*(\Omega).$$

Therefore, all closed sets Ω are Lebesgue-measurable. □

3.4 Characterization of Lebesgue-Measurable Sets

One way of characterizing Lebesgue-measurable sets is by the following theorem:

Inner-Outer Characterization of Measurability

A set Ω is Lebesgue-measurable in the sense of Section 3.1 if and only if for every $\varepsilon > 0$ there is an open set $\Lambda \supset \Omega$ and a closed set $\Xi \subset \Omega$ such that

$$m(\Lambda \backslash \Xi) < \varepsilon.$$

This characterization is different than the one presented in the Lecture Notes, which characterizes Lebesgue-measurable sets from the outside (i.e. outer characterization of measurability). However, these two characterizations leads to the same set of Lebesgue-measurable sets, so there is no contradiction whatsoever.

3.4.1 Proof in the Forward Direction

Suppose that Ω is Lebesgue-measurable in the sense of Section 3.1 with $m(\Omega) < \infty$. Then,

$$m(\Omega) = m^*(\Omega) = m_*(\Omega).$$

By definition of the Lebesgue outer measure of Ω, for any $\varepsilon > 0$ there is an open set $\Lambda \supset \Omega$ with

$$m(\Lambda) < m^*(\Omega) + \frac{\varepsilon}{2} = m(\Omega) + \frac{\varepsilon}{2}.$$

Similarly, by definition of the Lebesgue inner measure of Ω, there exists a compact (closed) set $\Xi \subset \Omega$ with

$$m(\Xi) \geq m_*(\Omega) - \frac{\varepsilon}{2} = m(\Omega) - \frac{\varepsilon}{2}.$$

Since Λ is open and Ξ is closed, $\Lambda \backslash \Xi$ is open. As all open sets and all closed sets are Lebesgue-measurable, then $\Lambda \backslash \Xi$ and Ξ are both Lebesgue-measurable and disjoint by construction. As such (by Section 3.2),

$$m(\Lambda \backslash \Xi) = m(\Lambda) - m(\Xi) < m(\Omega) + \frac{\varepsilon}{2} - \left(m(\Omega) - \frac{\varepsilon}{2}\right) = \varepsilon. \quad \square$$

On the other hand, if Ω is Lebesgue-measurable with $m(\Omega) = \infty$, we need to prove the statement for all $\Omega_i := \Omega \cap \text{box}[-a_i, a_i]$ for $(a_i)_j = i$ for all $i \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$. In this case, there exists open sets $\Lambda_i \supset \Omega_i$ and closed sets $\Xi_i \subset \Omega_i$ such that

$$m(\Omega_i \backslash \Xi_i) = m^*(\Omega_i \backslash \Xi_i) < \frac{\varepsilon}{2^i}.$$

Let $\Lambda := \cup_i \Lambda_i$ and

$$\Xi := \bigcup_i \left(\Xi_i \cap \left(\text{box}[-i, i] \backslash \text{box}[-(i-1), i-1]\right)\right).$$

As all (convergent) sequences in Ξ must eventually reach a point, and that point will be inside a box of side length one, then the point will belong to (at most) two entries in the union defining Ξ. Therefore, Ξ is closed.

Therefore, we have that $\Xi \subset \Omega \subset \Lambda$ and

$$\Lambda \backslash \Xi \subset \bigcup_i \left(\Lambda_i \backslash \Xi_i\right),$$

and by the fact that $\Lambda \backslash \Xi$ and $\Lambda_i \backslash \Xi_i$ are all open sets (and thus is Lebesgue-measurable), along with the monotonicity and subadditivity of the Lebesgue outer measure,

$$m(\Lambda \backslash \Xi) = m^*(\Lambda \backslash \Xi) \leq m^*\left(\bigcup_i (\Lambda_i \backslash \Xi_i)\right) \leq \sum_i m^*(\Lambda_i \backslash \Xi_i) < \sum_i \frac{\varepsilon}{2^i} = \varepsilon. \quad \square$$

3.4.2 Proof in the Backwards Direction

Assume that for every $\varepsilon > 0$, there exists Λ and Ξ such that $\Xi \subset \Omega \subset \Lambda$ with $m(\Lambda \backslash \Xi) < \varepsilon$.

First, consider Ω with $m^*(\Omega) < \infty$. Then, $m(\Xi) < \infty$, and

$$m^*(\Lambda) = m(\Lambda) = m(\Lambda \backslash \Xi) + m(\Xi) < \varepsilon + m(\Xi) < \infty.$$

Then, we have:

$$m^*(\Omega) \leq m(\Lambda) < m(\Xi) + \varepsilon = m_*(\Xi) + \varepsilon \leq m_*(\Omega) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that $m^*(\Omega) \leq m_*(\Omega)$. Combining this with the fact that $m_*(\Omega) \leq m^*(\Omega)$, we conclude that $m^*(\Omega) = m_*(\Omega)$, and Ω is Lebesgue-measurable. □

If Ω is Lebesgue-measurable and $m^*(\Omega) = \infty$, then we have (a_i satisfies $(a_i)_j = i$ for all $j \in \{1, \ldots, n\}$)

$$(\Xi \cap \text{box}[-a_i, a_i]) \subset (\Omega \cap \text{box}[-a_i, a_i]) \subset (\Lambda \cap I_i)\,,$$

where I_i is a box containing $\text{box}[-a_i, a_i]$ with $v(I_i) - v(\text{box}[-a_i, a_i]) < \varepsilon$. Thus,

$$m\left((\Lambda \cap I_i) \setminus (\Xi \cap \text{box}[-a_i, a_i])\right) < \varepsilon + \varepsilon = 2\varepsilon,$$

and the same argument with the $m^*(\Omega) < \infty$ case can be applied to conclude

$$m_*\left(\Omega \cap \text{box}[-a_i, a_i]\right) = m^*\left(\Omega \cap \text{box}[-a_i, a_i]\right).$$

Therefore, we conclude that Ω is Lebesgue-measurable in the sense of Section 3.1. □

3.5 Other Properties of Lebesgue-Measurable Sets

Union, Intersection, and Complement of Lebesgue-Measurable Sets

If $\{\Omega_j\}_j$ is a finite (or countable) collection of Lebesgue-measurable sets, then:
1) If Ω is Lebesgue-measurable, $\Omega^c := \mathbb{R}^n \backslash \Omega$ is also Lebesgue-measurable (**Theorem 2.2.10**).
2) If $\Omega := \cap_j \Omega_j$, then Ω is Lebesgue-measurable (**Exercise 2.2.8**, part 4).
3) If Ω_1 and Ω_2 are Lebesgue-measurable, then $\Omega_1 \backslash \Omega_2$ is also Lebesgue-measurable.
4) If $\Omega := \cup_j \Omega_j$, then Ω is Lebesgue-measurable with $m(\Omega) \leq \sum_j m(\Omega_j)$ (**Exercise 2.2.8**, part 3).

We will prove that these properties are natural consequences of the inner-outer characterization of Lebesgue-measurable sets:

1) If Ω is Lebesgue-measurable, then there exists Ξ and Λ such that $\Xi \subset \Omega \subset \Lambda$ and $m(\Lambda \backslash \Xi) < \varepsilon$. Notice that if $\Xi \subset \Omega \subset \Lambda$, then $\Xi^c \supset \Omega^c \supset \Lambda^c$ with Ξ^c open and Λ^c closed. Furthermore, $\Xi^c \backslash \Lambda^c = \Lambda \backslash \Xi$, so $m\left(\Xi^c \backslash \Lambda^c\right) = m(\Lambda \backslash \Xi) < \varepsilon$. Therefore, we conclude that Ω^c is Lebesgue-measurable. □
2) For any $\varepsilon > 0$ and all j (note that $j \in \mathbb{N}$) choose Λ_j open and Ξ_j closed such that $\Xi_j \subset \Omega_j \subset \Lambda_j$ and

$$m(\Lambda_j \backslash \Xi_j) < \frac{\varepsilon}{2^j}.$$

Then, we have that:

$$\bigcap_j \Xi_j \subset \bigcap_j \Omega_j \subset \bigcap_j \Lambda_j.$$

As a consequence, we have:

$$\Big(\bigcap_j \Lambda_j\Big) \backslash \Big(\bigcap_j \Xi_j\Big) \subset \bigcup_j \left(\Lambda_j \backslash \Xi_j\right).$$

As both the left-hand and right-hand sets are open, they are Lebesgue-measurable. Therefore,

$$m\left(\Big(\bigcap_j \Lambda_j\Big)\backslash\Big(\bigcap_j \Xi_j\Big)\right) \leq m^*\left(\bigcup_j (\Lambda_j\backslash\Xi_j)\right) \leq \sum_j m^*(\Lambda_j\backslash\Xi_j) = \sum_j m(\Lambda_j\backslash\Xi_j) < \sum_j \frac{\varepsilon}{2^j} < \varepsilon.$$

As such, we conclude that $\cap_j \Omega_j$ is Lebesgue-measurable. □

3) We rewrite $\Omega_1\backslash\Omega_2$ as:

$$\Omega_1\backslash\Omega_2 = \Omega_1 \cup \Omega_2^c.$$

Since the complement of a Lebesgue-measurable set is Lebesgue-measurable, and the intersection between two Lebesgue-measurable sets is also Lebesgue-measurable, then $\Omega_1\backslash\Omega_2$ is Lebesgue-measurable. □

As an aside, its Lebesgue measure is

$$m(\Omega_1\backslash\Omega_2) = m(\Omega_1) - m(\Omega_2).$$

4) We rewrite $\Omega = \cup_j \Omega_j$ as:

$$\Omega = \bigcup_j \Omega_j = \Big(\bigcap_j \Omega_j^c\Big)^c.$$

As the (finite or countable) intersection of Lebesgue-measurable sets is Lebesgue-measurable, and the complement of Lebesgue-measurable sets are also Lebesgue-measurable, then Ω (the complement of the intersection of countable collection of complements of such sets) is also Lebesgue-measurable.

Also observe that:

$$\Omega = \bigcup_j \Omega_j = \bigcup_j \Big(\Omega_j\backslash \bigcup_{k<j} (\Omega_k \cap \Omega_j)\Big).$$

Since the measure of a set is never negative, we have that:

$$m(\Omega) = \sum_j m(\Omega)_j - m(\cap) \leq \sum_j m(\Omega_j), \quad \square$$

where $m(\cap)$ is the Lebesgue measure of the union of intersections part.