On the Convergence of Distributions

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This document will show the method of proving the convergence of a sequence of distributions $(T_j)_{j\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ to a certain distribution $T_\infty \in \mathcal{D}'(\mathbb{R}^n)$. To this end, the proofs of **Exercise 1.3.8.** will be provided, and a result essential to **Exercise 1.3.9** will be given. First, the definition of convergence in $\mathcal{D}'(\mathbb{R}^n)$:

Definition 1.3.4 (Convergence in $\mathcal{D}'(\mathbb{R}^n)$)

A sequence $(T_j)_{j\in\mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ of distributions converges to $T_\infty \in \mathcal{D}'(\mathbb{R}^n)$ if $\lim_{j\to\infty} T_j(f) = T_\infty(f)$ for all $f \in \mathcal{D}(\mathbb{R}^n)$. In this case, we write $T_j \to T_\infty$ in $\mathcal{D}'(\mathbb{R}^n)$ as $j \to \infty$.

1 Proof of Exercise 1.3.8

Exercise 1.3.8

Consider $h : \mathbb{R}^n \to \mathbb{K}$ satisfying $\int_{\mathbb{R}^n} |h(X)|\ \mathrm{d}X < \infty$, and assume that $\int_{\mathbb{R}^n} h(X)\,\mathrm{d}X = 1$. For $j \in \mathbb{N}$, set $h_j(X) := j^n h(jX)$. then, prove that $T_{h_j} \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $j \to \infty$. Equivalently, for $\varepsilon > 0$ one often sets $h_\varepsilon(X) := \frac{1}{\varepsilon^n} h\left(\frac{X}{\varepsilon}\right)$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.

While the two statements are generally equivalent, both can be proven independently.

1.1 Proof of convergence of T_{h_j} to δ_0 in $\mathcal{D}'(\mathbb{R}^n)$ as $j \to \infty$

Let $\varepsilon > 0$ (for the purposes of proving convergence) be arbitrarily given.

Observe that:

$$1 = \int h(X)\,\mathrm{d}X \leq \int |h(X)|\ \mathrm{d}X < \infty.$$

As such, for any $f \in \mathcal{D}(\mathbb{R}^n)$, fix $r > 0$ $\big($and set $K_r := \mathbb{R}^n \backslash \mathcal{B}_r(0)\big)$ such that

$$\|f\|_\infty \int_{K_r} |h(X)|\ \mathrm{d}X < \frac{\varepsilon}{4}. \tag{♥}$$

f is a test function, so it is continuous. Therefore, there exists J such that for any $j \geq J$ and r fixed in (♥):

$$\sup_{U\in\mathcal{B}_r(0)} \left| f\left(\frac{U}{j}\right) - f(0)\right| \leq \frac{\varepsilon}{2\int_{\mathcal{B}_r(0)} |h(X)|\ \mathrm{d}X}. \tag{♥♥}$$

By definition of T_h and $h_j(X)$, and setting $U := jX$ one has:

$$T_{h_j}(f) = \int_{\mathbb{R}^n} j^n h(jX) f(X)\,\mathrm{d}X = \int_{\mathbb{R}^n} h(U) f\left(\frac{U}{j}\right)\,\mathrm{d}U.$$

For r and J fixed in (♥) and (♥♥), respectively, one has:

$$\begin{aligned}\left|T_{h_j}(f) - f(0)\right| &= \left| \int_{\mathbb{R}^n} f\left(\frac{U}{j}\right) h(U)\ \mathrm{d}U - \int_{\mathbb{R}^n} f(0) h(U)\ \mathrm{d}U \right| \\ &= \left| \int_{\mathcal{B}_r(0)} h(U)\left[f\left(\frac{U}{j}\right) - f(0)\right]\ \mathrm{d}U - \int_{K_r} h(U)\left[f\left(\frac{U}{j}\right) - f(0)\right]\ \mathrm{d}U \right|\end{aligned}$$

$$\leq \left|\int_{\mathcal{B}_r(0)} h(U)\left[f\left(\frac{U}{j}\right)-f(0)\right]\,\mathrm{d}U\right| + \left|\int_{K_r} h(U)\left[f\left(\frac{U}{j}\right)-f(0)\right]\,\mathrm{d}U\right|$$
$$\leq \sup_{U\in\mathcal{B}_r(0)}\left|f\left(\frac{U}{j}\right)-f(0)\right|\cdot\int_{\mathcal{B}_r(0)}|h(X)|\,\mathrm{d}X + 2\left\|f\right\|_\infty\int_{K_r}|h(X)|\,\mathrm{d}X.$$

By the choice of r and J above, one has for $j \geq J$:

$$\left|T_{h_j}(f)-f(0)\right| \leq \frac{\varepsilon}{2\int_{\mathcal{B}_r(0)}|h(X)|\,\mathrm{d}X}\cdot\int_{\mathcal{B}_r(0)}|h(X)|\,\mathrm{d}X + 2\cdot\frac{\varepsilon}{4} = \varepsilon.$$

This is exactly the definition of convergence of $T_{h_j}(f) \to f(0)$ in $\mathbb{K}$ as $j\to\infty$ and for any $f\in\mathcal{D}(\mathbb{R}^n)$. As such, by the definition of convergence of distributions, one concludes that $T_{h_j}\to\delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $j\to\infty$. □

1.2 Proof of convergence of T_{h_ε} to δ_0 in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \searrow 0$

The proof of this statement is nearly identical to the previous statement, with minimal differences. Therefore, this subsection can be skipped if deemed to be not necessary.

Let $\varepsilon' > 0$ (for the purposes of proving convergence) be arbitrarily given.

Observe that:

$$1 = \int h(X)\,\mathrm{d}X \leq \int |h(X)|\,\mathrm{d}X < \infty.$$

As such, for any $f\in\mathcal{D}(\mathbb{R}^n)$, fix $r>0$ $\big($and set $K_r := \mathbb{R}^n\backslash\mathcal{B}_r(0)\big)$ such that

$$\|f\|_\infty\int_{K_r}|h(X)|\,\mathrm{d}X < \frac{\varepsilon'}{4}. \qquad (♥)$$

f is a test function, so it is continuous. Therefore, there exists ε_0 such that for the fixed r and any $\varepsilon\leq\varepsilon_0$,

$$\sup_{U\in\mathcal{B}_r(0)}\left|f\left(U\varepsilon\right)-f(0)\right| \leq \frac{\varepsilon'}{2\int_{\mathcal{B}_r(0)}|h(X)|\,\mathrm{d}X}. \qquad (♥♥)$$

By definition of T_h and $h_\varepsilon(X)$, and setting $U := \frac{X}{\varepsilon}$ one has:

$$T_{h_\varepsilon}(f) = \int_{\mathbb{R}^n}\frac{1}{\varepsilon^n}h\left(\frac{X}{\varepsilon}\right)f(X)\,\mathrm{d}X = \int_{\mathbb{R}^n}h(U)f\left(U\varepsilon\right)\,\mathrm{d}U.$$

For r and ε_0 fixed in (♥) and (♥♥), respectively, one has:

$$\left|T_{h_\varepsilon}(f)-f(0)\right| = \left|\int_{\mathbb{R}^n}f\left(U\varepsilon\right)h(U)\,\mathrm{d}U - \int_{\mathbb{R}^n}f(0)h(U)\,\mathrm{d}U\right|$$
$$= \left|\int_{\mathcal{B}_r(0)}h(U)\left[f\left(U\varepsilon\right)-f(0)\right]\,\mathrm{d}U - \int_{K_r}h(U)\left[f\left(U\varepsilon\right)-f(0)\right]\,\mathrm{d}U\right|$$
$$\leq \left|\int_{\mathcal{B}_r(0)}h(U)\left[f\left(U\varepsilon\right)-f(0)\right]\,\mathrm{d}U\right| + \left|\int_{K_r}h(U)\left[f\left(U_\varepsilon\right)-f(0)\right]\,\mathrm{d}U\right|$$
$$\leq \sup_{U\in\mathcal{B}_r(0)}\left|f\left(U\varepsilon\right)-f(0)\right|\cdot\int_{\mathcal{B}_r(0)}|h(X)|\,\mathrm{d}X + 2\left\|f\right\|_\infty\int_{K_r}|h(X)|\,\mathrm{d}X.$$

By the choice of r and ε_0 above, one has for $\varepsilon\leq\varepsilon_0$:

$$\left|T_{h_\varepsilon}(f)-f(0)\right| \leq \frac{\varepsilon'}{2\int_{\mathcal{B}_r(0)}|h(X)|\,\mathrm{d}X}\cdot\int_{\mathcal{B}_r(0)}|h(X)|\,\mathrm{d}X + 2\cdot\frac{\varepsilon'}{4} = \varepsilon'.$$

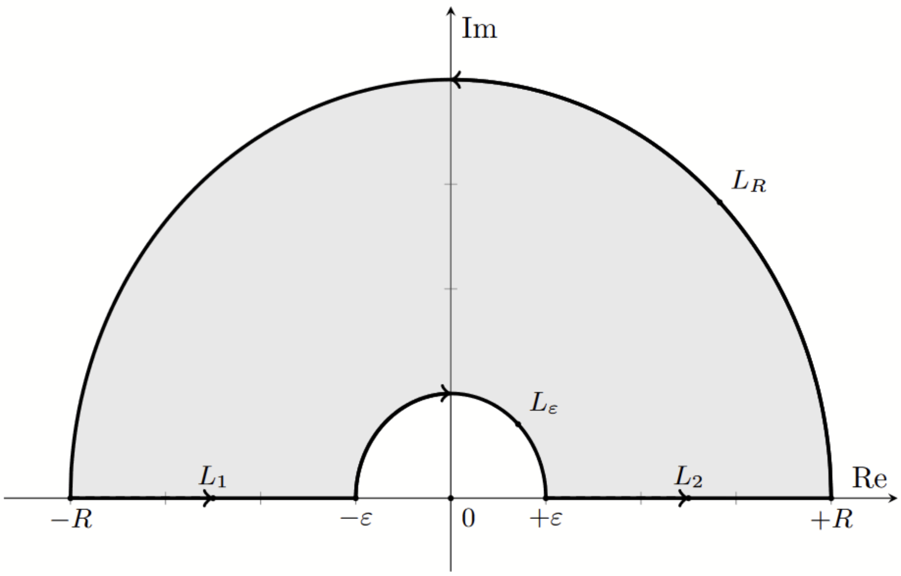
This is exactly the definition of convergence of $T_{h_\varepsilon}(f)\to f(0)$ in $\mathbb{K}$ as $\varepsilon\searrow 0$ and for any $f\in\mathcal{D}(\mathbb{R}^n)$. As such, by the definition of convergence of distributions, one concludes that $T_{h_\varepsilon}\to\delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon\searrow 0$. □

2 Evaluating the Improper Integral $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, \mathrm{d}x$

This result is essential to Exercise 1.3.9. (although, the proof of said exercise is not included in this document).

This proof uses elements of complex analysis. For those not familiar with complex analysis, please look up the required terms and/or results if necessary.

Consider a contour C in the complex plane formed by a semicircle of radius R, albeit with a smaller semicircle of radius ε taken out from it, as illustrated in the picture below:



Mathematically, it can be written as the union of four paths, defined by:

$$\begin{aligned} L_\varepsilon &= \left\{Re^{it} \mid 0 \le t \le \pi\right\}, \\ L_1 &= \{t \mid -R \le t \le -\varepsilon\}, \\ L_R &= \left\{\varepsilon e^{-it} \mid \pi \le t \le 2\pi\right\}, \\ L_2 &= \{t \mid \varepsilon \le t \le R\}. \end{aligned}$$

As such, one writes

$$C = L_\varepsilon \cup L_2 \cup L_R \cup L_1.$$

One important result in Complex Analysis is:

> **Cauchy's Theorem** (also called **Cauchy's Integral Theorem**)
>
> ---
>
> For any function f that is holomorphic in a simply connected domain $\Omega \subseteq \mathbb{C}$ and any closed contour C in Ω, the contour integral of f along C is zero. In mathematical terms,
>
> $$\oint_C f(z) \, \mathrm{d}z = 0$$

Set Ω to be the closed subset of $\mathbb{C}$ that is bounded by (and including) the (closed) contour C. As Ω can be compressed to a single point, it is simply connected. Then, consider a function $\mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \frac{e^{iz}}{z}.$$

This function has a singularity at $z = 0$ (i.e. has a pole at $z = 0$, so is not well-defined), but is well-defined and continuous elsewhere. By construction, Ω does not contain the point $z = 0$, so f is holomorphic on Ω.

Therefore, by **Cauchy's Theorem**,

$$\oint_C f(z)\,\mathrm{d}z = \int_{L_\varepsilon} f(z)\,\mathrm{d}z + \int_{L_2} f(z)\,\mathrm{d}z + \int_{L_R} f(z)\,\mathrm{d}z + \int_{L_1} f(z)\,\mathrm{d}z = 0 \tag{Ca}$$

Note that the entire real line $\mathbb{R}$ can be viewed as the limit of the union of L_1 and L_2 as $\varepsilon \searrow 0$ and $R \to \infty$.

First, consider the modulus of the integral of $f(z)$ along L_R (use substitution $z = Re^{it}$ along L_R):

$$\begin{aligned}\left|\int_{L_R} \frac{f(z)}{z}\,\mathrm{d}z\right| &= \left|\int_0^\pi \frac{\exp\left(iRe^{it}\right)}{Re^{it}} \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t\right| \\ &= \left|\int_0^\pi \frac{\exp\left(i\cdot(R\cos t + i\sin t)\right)}{Re^{it}} \cdot iRe^{it}\,\mathrm{d}t\right| \\ &= \left|i\int_0^\pi \exp\left(iR\cos t\right)\exp\left(-R\sin t\right)\,\mathrm{d}t\right|.\end{aligned}$$

Since any number of the form $\exp(ix)$ has modulus 1 (and that i is also of modulus 1), one has:

$$\left|\int_{L_R} \frac{f(z)}{z}\,\mathrm{d}z\right| = \int_0^\pi \exp\left(-R\sin t\right)\,\mathrm{d}t = 2\int_0^{\pi/2} \exp\left(-R\sin t\right)\,\mathrm{d}t.$$

The modulus value can be removed since this is now simply an integral of one (real) variable, and is always positive over the values of t (exp is a strictly positive function). The second equality is due to the trigonometric identity $\sin(t) = \sin\left(\frac{\pi}{2} - t\right)$. By observing that for $0 \le t \le \frac{\pi}{2}$, $\sin(t) \ge \frac{2t}{\pi}$ and setting $u = \frac{-2Rt}{\pi}$, one obtains:

$$\begin{aligned}\left|\int_{L_R} \frac{f(z)}{z}\,\mathrm{d}z\right| &\le \int_0^{\pi/2} \exp\left(-\frac{2Rt}{\pi}\right)\,\mathrm{d}t = 2\int_0^{-R} \frac{-\pi}{2R}\exp(u)\,\mathrm{d}u \\ &= \frac{\pi}{R}\int_{-R}^0 \exp(u)\,\mathrm{d}u = \frac{\pi}{R}\left(1 - e^{-R}\right).\end{aligned}$$

Taking the limit as $R \to \infty$ one observes that:

$$\lim_{R\to\infty}\left|\int_{L_R} \frac{f(z)}{z}\,\mathrm{d}z\right| \le \lim_{R\to\infty} \frac{\pi}{R}\left(1 - e^{-R}\right) = 0.$$

The only number in $\mathbb{C}$ with modulus 0 is 0, so one concludes that the integral along L_R goes to 0 as $R \to \infty$.

Next, consider the integral of $f(z)$ along L_ε (make the substitution $z = \varepsilon e^{it}$):

$$\int_{L_\varepsilon} \frac{f(z)}{z}\,\mathrm{d}z = \int_{\pi}^{2\pi} \frac{\exp\left(i\varepsilon e^{-it}\right)}{\varepsilon e^{-it}} \cdot -i\varepsilon e^{-it}\,\mathrm{d}t = -i\int_{\pi}^{2\pi} \exp\left(i\varepsilon e^{-it}\right)\,\mathrm{d}t.$$

Taking the limit as $\varepsilon \searrow 0$ and by an application of the dominated convergence theorem, one obtains:

$$\lim_{\varepsilon\searrow 0}\int_{L_\varepsilon} \frac{f(z)}{z}\,\mathrm{d}z = -i\int_{\pi}^{2\pi} \lim_{\varepsilon\searrow 0} \exp\left(-\varepsilon e^{it}\right)\,\mathrm{d}t = -i\int_{\pi}^{2\pi} 1\,\mathrm{d}t = -i\pi.$$

Taking the limit as $\varepsilon \searrow 0$ and $R \to \infty$ of Equation (Ca) one obtains:

$$\lim_{\substack{\varepsilon\searrow 0\\ R\to\infty}} \left[\int_{L_2} f(z)\,\mathrm{d}z + \int_{L_1} f(z)\,\mathrm{d}z\right] - i\pi = 0.$$

For z in the real line $\mathbb{R}$, one can write the above equation as:

$$\lim_{\substack{\varepsilon\searrow 0\\ R\to\infty}} \left[\int_{-R}^{-\varepsilon} \frac{\cos(z) + i\sin(z)}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{\cos(z) + i\sin(z)}{z}\,\mathrm{d}z\right] = i\pi.$$

Separating the real and imaginary parts one obtains the following two equations:

$$\lim_{\substack{\varepsilon\searrow 0\\ R\to\infty}} \left[\int_{-R}^{-\varepsilon} \frac{\cos(z)}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{\cos(z)}{z}\,\mathrm{d}z\right] = 0.$$

$$\lim_{\substack{\varepsilon\searrow 0\\ R\to\infty}} \left[\int_{-R}^{-\varepsilon} \frac{\sin(z)}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{\sin(z)}{z}\,\mathrm{d}z\right] = \pi.$$

The first equality is clear, since an odd function is integrated over a symmetric domain.

From the second one, one obtains the desired result:

$$\int_{\mathbb{R}} \frac{\sin(x)}{x}\,\mathrm{d}x = \pi. \quad \square$$

Here, this integral can be understood in the sense of an improper Riemann integral.