Fourier transform of some important tempered distributions

SML course - Introduction to functional analysis

NGUYEN Tue Tai / 062201848

May 9, 2023

1 Proof that $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$

Let $p \in [1, \infty)$ and define

$$L^p(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{K} \;\middle|\; \int_{\mathbb{R}^n} |f(X)|^p \, \mathrm{d}X < \infty \right\}.$$

Observe that

$$\int_{\mathbb{R}} \frac{1}{(1+x^2)^p} \, \mathrm{d}x \leq \int_{\mathbb{R}} \frac{1}{1+x^2} \, \mathrm{d}x = \pi < \infty.$$

Let $\phi : \mathbb{R}^n \to \mathbb{K}$ be defined by

$$\phi(X) = \frac{1}{(1+x_1^2)(1+x_2^2)\dots(1+x_n^2)} = \frac{1}{\prod_{i=1}^{n}(1+x_i^2)}.$$

We have $\phi \in L^p(\mathbb{R}^n)$ for any $p \in [1, \infty)$ because

$$\int_{\mathbb{R}^n} |\phi(X)|^p \, \mathrm{d}X = \int_{\mathbb{R}} \frac{1}{(1+x_1^2)^p} \, \mathrm{d}x_1 \int_{\mathbb{R}} \frac{1}{(1+x_2^2)^p} \, \mathrm{d}x_2 \dots \int_{\mathbb{R}} \frac{1}{(1+x_n^2)^p} \, \mathrm{d}x_n < \infty.$$

Letting $\gamma = (2, 2, \dots, 2) \in \mathbb{N}^n$, we have for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $X \in \mathbb{R}^n$,

$$\begin{aligned}
\frac{|f(X)|}{|\phi(X)|} = \left|(1+x_1^2)(1+x_2^2)\dots(1+x_n^2) f(X)\right| &= \left| \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \dots \sum_{i_n=0}^{1} x_1^{2i_1} x_2^{2i_2} \dots x_n^{2i_n} f(X) \right| \\
&\leq \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} \dots \sum_{i_n=0}^{1} \left| x_1^{2i_1} x_2^{2i_2} \dots x_n^{2i_n} f(X) \right| \\
&\leq \sum_{|\beta| \leq |\gamma|} |X^\beta f(X)| \\
&\leq \sum_{|\beta| \leq |\gamma|} \left\| X^\beta f \right\|_\infty < \infty
\end{aligned}$$

Hence, we have for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(X)|^p \, \mathrm{d}X = \int_{\mathbb{R}^n} \frac{|f(X)|^p}{|\phi(X)|^p} |\phi(X)|^p \, \mathrm{d}X &\leq \int_{\mathbb{R}^n} \left(\sum_{|\beta| \leq |\gamma|} \left\| X^\beta f \right\|_\infty \right)^p |\phi(X)|^p \, \mathrm{d}X \\
&= \left(\sum_{|\beta| \leq |\gamma|} \left\| X^\beta f \right\|_\infty \right)^p \int_{\mathbb{R}^n} |\phi(X)|^p \, \mathrm{d}X < \infty.
\end{aligned}$$

Thus, if $f \in \mathcal{S}(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$. As a result, $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$.

2 Proof that $T_{\lambda\mathbf{1}}$ is a tempered distribution

Consider the function $\mathbf{1} : \mathbb{R}^n \to \mathbb{K}$ defined by $\mathbf{1}(X) = 1$ for $X \in \mathbb{R}^n$. Let $\lambda \in \mathbb{K}$ and define $T_{\lambda\mathbf{1}} : \mathcal{S}(\mathbb{R}^n) \to \mathbb{K}$ by

$$T_{\lambda\mathbf{1}}(f) = \int_{\mathbb{R}^n} \lambda\mathbf{1}(X)f(X)\,\mathrm{d}X = \lambda \int_{\mathbb{R}^n} f(X)\,\mathrm{d}X\,,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Indeed, $T_{\lambda\mathbf{1}}$ is well-defined because $f \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Now, we want to show that $T_{\lambda\mathbf{1}}$ is a tempered distribution. Clearly, $T_{\lambda\mathbf{1}}$ is linear, so we just need to prove that $T_{\lambda\mathbf{1}}$ is continuous.

Using ϕ from Section 1 with $\phi \in L^1(\mathbb{R}^n)$ and $\gamma = (2, 2, \dots, 2) \in \mathbb{N}^n$, we have for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
|T_{\lambda\mathbf{1}}(f)| = \left|\lambda \int_{\mathbb{R}^n} f(X)\,\mathrm{d}X\right| \le |\lambda| \int_{\mathbb{R}^n} |f(X)|\,\mathrm{d}X &= |\lambda| \int_{\mathbb{R}^n} |\phi(X)| \frac{|f(X)|}{|\phi(X)|}\,\mathrm{d}X \\
&\le |\lambda| \int_{\mathbb{R}^n} |\phi(X)| \sum_{|\beta|\le|\gamma|} \left\|X^\beta f\right\|_\infty \mathrm{d}X \\
&= \left(\int_{\mathbb{R}^n} |\lambda\phi(X)|\,\mathrm{d}X\right) \sum_{|\beta|\le|\gamma|} \left\|X^\beta f\right\|_\infty \\
&\le c \sum_{|\alpha|,|\beta|\le|\gamma|} \left\|X^\beta \partial^\alpha f\right\|_\infty,
\end{aligned}$$

where $c = \int_{\mathbb{R}^n} |\lambda\phi(X)|\,\mathrm{d}X$.

By **Theorem 1.5.10**, $T_{\lambda\mathbf{1}}$ is continuous and it is a tempered distribution.

3 Proof that δ_0 is a tempered distribution

Define $\delta_0 : \mathcal{S}(\mathbb{R}^n) \to \mathbb{K}$ by $\delta_0(f) = f(0)$ for $f \in \mathcal{S}(\mathbb{R}^n)$. We have for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathbb{N}$,

$$|\delta_0(f)| = |f(0)| \le \|f\|_\infty \le 1 \cdot \sum_{|\alpha|,|\beta|\le m} \left\|X^\beta \partial^\alpha f\right\|_\infty.$$

Because δ_0 is also linear, δ_0 is a tempered distribution by **Theorem 1.5.10**.

4 Proof that T_h is a tempered distribution for any $h \in L^p(\mathbb{R}^n)$

Let $h \in L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$ and define $T_h : \mathcal{S}(\mathbb{R}^n) \to \mathbb{K}$ by

$$T_h(f) = \int_{\mathbb{R}^n} h(X)f(X)\,\mathrm{d}X \qquad \text{for any } f \in \mathcal{S}(\mathbb{R}^n).$$

This expression is well-defined because for any $f \in \mathcal{S}(\mathbb{R}^n) \subset L^q$ with $q \in [1, \infty)$ and $1/p + 1/q = 1$,

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} h(X)f(X)\,\mathrm{d}X\right| \le \int_{\mathbb{R}^n} |h(X)f(X)|\,\mathrm{d}X &\le \left(\int_{\mathbb{R}^n} |h(X)|^p\,\mathrm{d}X\right)^{1/p} \left(\int_{\mathbb{R}^n} |f(X)|^q\,\mathrm{d}X\right)^{1/q} \\
&= \|h\|_p \|f\|_q < \infty,
\end{aligned}$$

where

$$\|h\|_p = \left(\int_{\mathbb{R}^n} |h(X)|^p\,\mathrm{d}X\right)^{1/p}, \qquad \|f\|_q = \left(\int_{\mathbb{R}^n} |f(X)|^q\,\mathrm{d}X\right)^{1/q}.$$

The second inequality is called Hölder's inequality. When $p = 1$, we can replace $\|f\|_q$ by $\|f\|_\infty$ (which is finite due to the definition of Schwartz functions), so the argument is still valid for any $p \in [1, \infty)$.

Using ϕ from Section 1 with $\phi \in L^q(\mathbb{R}^n)$ and $\gamma = (2, 2, \ldots, 2) \in \mathbb{N}^n$, we have for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\left| \int_{\mathbb{R}^n} h(X) f(X) \, \mathrm{d}X \right| &\leq \int_{\mathbb{R}^n} |h(X) f(X)| \, \mathrm{d}X \\
&= \int_{\mathbb{R}^n} |h(X)\phi(X)| \frac{|f(X)|}{|\phi(X)|} \, \mathrm{d}X \\
&\leq \int_{\mathbb{R}^n} |h(X)\phi(X)| \sum_{|\beta| \leq |\gamma|} \left\| X^\beta f \right\|_\infty \mathrm{d}X \\
&= \left(\int_{\mathbb{R}^n} |h(X)\phi(X)| \, \mathrm{d}X \right) \sum_{|\beta| \leq |\gamma|} \left\| X^\beta f \right\|_\infty \\
&\leq \|h\|_p \|\phi\|_q \sum_{|\alpha|, |\beta| \leq |\gamma|} \left\| X^\beta \partial^\alpha f \right\|_\infty \\
&= c \sum_{|\alpha|, |\beta| \leq |\gamma|} \left\| X^\beta \partial^\alpha f \right\|_\infty ,
\end{aligned}$$

where $c = \|h\|_p \|\phi\|_q$. Notice that $\mathrm{Ran}(\phi) = (0, 1]$ so $\|\phi\|_\infty < \infty$ when $q = \infty$.

Because T_h is also linear, T_h is a tempered distribution by **Theorem 1.5.10** for any $h \in L^p(\mathbb{R}^n)$ and $p \in [1, \infty)$.

5 Exercise 1.5.12

1. We have for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
[\mathcal{F}\,\delta_0](f) = \delta_0(\mathcal{F}\,f) = [\mathcal{F}\,f](0) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i0\cdot X} f(X) \, \mathrm{d}X \\
&= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} f(X) \, \mathrm{d}X \\
&= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \mathbf{1}(X) f(X) \, \mathrm{d}X \\
&= T_{(2\pi)^{-n/2}\mathbf{1}}(f).
\end{aligned}$$

   Hence, $\mathcal{F}\,\delta_0 = T_{(2\pi)^{-n/2}\mathbf{1}}$.

2. For any $f \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
[\mathcal{F}\,T_{\mathbf{1}}](f) = T_{\mathbf{1}}(\mathcal{F}\,f) &= \int_{\mathbb{R}^n} \mathbf{1}(X) [\mathcal{F}\,f](X) \, \mathrm{d}X \\
&= \int_{\mathbb{R}^n} [\mathcal{F}\,f](X) \, \mathrm{d}X \\
&= (2\pi)^{n/2} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i0\cdot X} [\mathcal{F}\,f](X) \, \mathrm{d}X \\
&= (2\pi)^{n/2} \left[\mathcal{F}^{-1}(\mathcal{F}\,f) \right](0) \\
&= (2\pi)^{n/2} f(0) \\
&= (2\pi)^{n/2} \delta_0(f).
\end{aligned}$$

   In the above calculation, we use the identity $\mathcal{F}^{-1}(\mathcal{F}\,f) = f$ due to the fact that Fourier transform is an isomorphism on Schwartz space $\mathcal{S}(\mathbb{R}^n)$. As a result, $\mathcal{F}\,T_{\mathbf{1}} = (2\pi)^{n/2}\delta_0$.

3. Let $h \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. We have for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}\,T_h(f) = T_h(\mathcal{F}\,f) = \int_{\mathbb{R}^n} h(\xi) [\mathcal{F}\,f](\xi) \, \mathrm{d}\xi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} h(\xi) \int_{\mathbb{R}^n} e^{-i\xi\cdot X} f(X) \, \mathrm{d}X \, \mathrm{d}\xi \, .$$

   Because $h \in L^1(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |h(\xi) e^{-i\xi\cdot X} f(X)| \,\mathrm{d}X \,\mathrm{d}\xi = \int_{\mathbb{R}^n} |h(\xi)| \,\mathrm{d}\xi \int_{\mathbb{R}^n} |f(X)| \,\mathrm{d}X < \infty.$$

Hence, we can apply the Fubini theorem:

$$\begin{aligned} \mathcal{F} T_h(f) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} h(\xi) \int_{\mathbb{R}^n} e^{-i\xi\cdot X} f(X) \,\mathrm{d}X \,\mathrm{d}\xi &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(X) \int_{\mathbb{R}^n} e^{-i\xi\cdot X} h(\xi) \,\mathrm{d}\xi \,\mathrm{d}X \\ &= \int_{\mathbb{R}^n} f(X) \hat{h}(X) \,\mathrm{d}X \,. \end{aligned}$$

As a result, $\mathcal{F} T_h = T_{\hat{h}}$ for any $h \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

To extend this result to $h \in L^2(\mathbb{R}^n)$, we need to use the fact that $\mathcal{F}$ is a unitary map on $L^2(\mathbb{R}^n)$, i.e. given the inner product $\langle \cdot, \cdot \rangle : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to \mathbb{K}$ defined by

$$\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(X)} g(X) \,\mathrm{d}X \,,$$

we have

$$\langle f, \mathcal{F} g \rangle = \langle \mathcal{F}^{-1} f, g \rangle \,.$$

Using this information, we have for $h \in L^2(\mathbb{R}^n)$ and any $f \in \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$,

$$\mathcal{F} T_h(f) = \int_{\mathbb{R}^n} h(X) [\mathcal{F} f](X) \,\mathrm{d}X = \langle \overline{h}, \mathcal{F} f \rangle = \langle \mathcal{F}^{-1} \overline{h}, f \rangle = \int_{\mathbb{R}^n} \overline{\left[\mathcal{F}^{-1} \overline{h}\right](X)} f(X) \,\mathrm{d}X \,,$$

where

$$\overline{\left[\mathcal{F}^{-1} \overline{h}\right](X)} = \overline{\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot X} \overline{h(\xi)} \,\mathrm{d}\xi} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot X} h(\xi) \,\mathrm{d}\xi = \hat{h}(X).$$

Hence,

$$\mathcal{F} T_h(f) = \int_{\mathbb{R}^n} \hat{h}(X) f(X) \,\mathrm{d}X = T_{\hat{h}}(f).$$

Therefore, $\mathcal{F} T_h = T_{\hat{h}}$ for any $h \in L^2(\mathbb{R}^n)$.

Comments

In physics, the results from 1. and 2. are usually written as (in 1-dimension)

$$[\mathcal{F} \delta](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \delta(x) \,\mathrm{d}x = \frac{1}{\sqrt{2\pi}},$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} \,\mathrm{d}x = \sqrt{2\pi} \delta(\xi).$$

Especially, the second result is usually rewritten as

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \,\mathrm{d}\xi \,,$$

which is sometimes treated as a "definition" of the delta Dirac funtion δ.

Besides, the results from 1. and 2. are useful mathematical relations that reflect the relationship between the position space and the momentum space in quantum mechanics.

Appendix

Hölder's inequality

For any $f \in L^p(\mathbb{R}^n)$ with $p \in [1, \infty)$, define $\|\cdot\|_p$ by

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(X)|^p \, \mathrm{d}X \right)^{1/p}.$$

For any $p, q \in [1, \infty)$ such that $1/p + 1/q = 1$, let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Then,

$$\int_{\mathbb{R}^n} |f(X)g(X)| \, \mathrm{d}X \leq \left(\int_{\mathbb{R}^n} |f(X)|^p \, \mathrm{d}X \right)^{1/p} \left(\int_{\mathbb{R}^n} |g(X)|^q \, \mathrm{d}X \right)^{1/q},$$

or equivalently,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Additionally, if $p = 1$ and $q = \infty$, then

$$\|fg\|_1 \leq \|f\|_1 \|g\|_\infty.$$