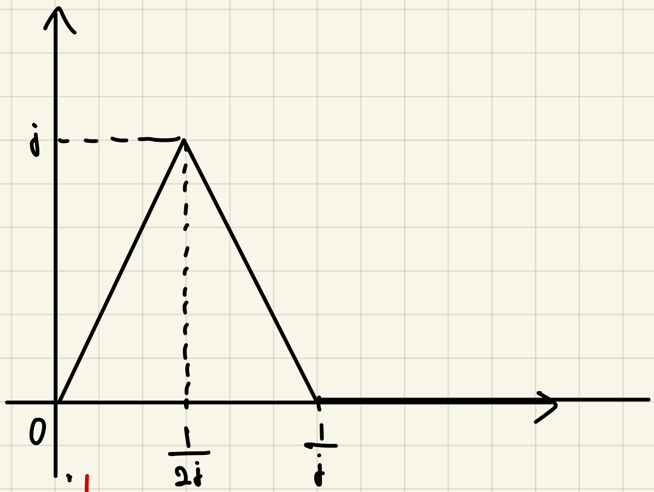
Ex 2.6.6

$x \in [0,1]$, consider the functions,

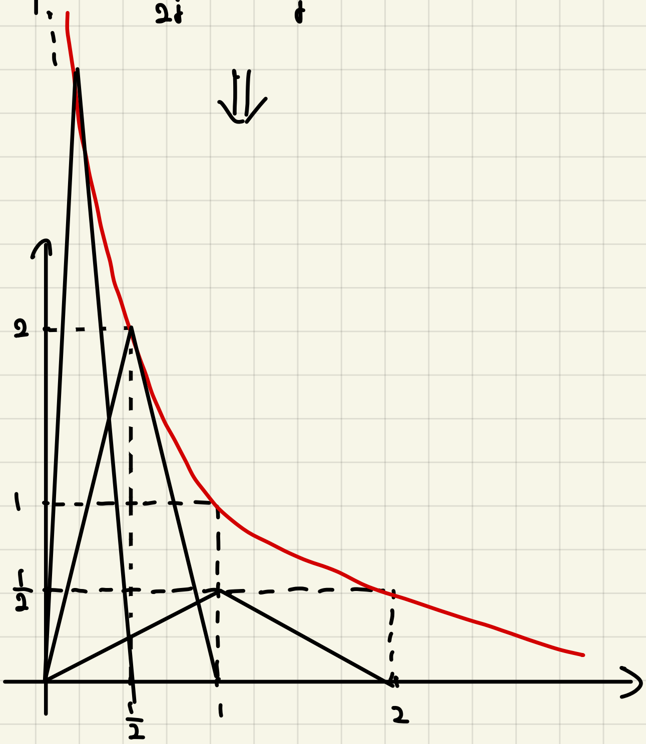
$$f_j(x) = \begin{cases} 2j^2 x & (\text{if } 0 \le x \le \frac{1}{2j}) \\ -2j^2\left(x - \frac{1}{j}\right) & (\text{if } \frac{1}{2j} < x \le \frac{1}{j}) \\ 0 & (\text{if } \frac{1}{j} < x \le 1) \end{cases}$$

First, check that $f_\infty = 0$.

Thinking about the graph,



$\Downarrow$



If $x = 0$,

$f(x) = 0$, so $\lim\limits_{j \to \infty} f_j = 0$

$\therefore f_\infty(0) = 0$ — ①

If $x \ne 0$, $x \in (0, 1]$,

for $1/j < x$, $f_j(x) = 0$. by def.

Thus,

we set $\forall N > \frac{1}{x}$ $\left(x > \frac{1}{N}\right)$

$\forall j \ge N$

such that $f_j(x) = 0$

Thus,

if we fix $x \in (0, 1]$,

$\lim\limits_{j \to \infty} f_j(x) = 0$ — ②

By ① & ②, for any $x \in [0, 1]$,

$\lim\limits_{j \to \infty} f_j(x) = 0$

$\therefore f_\infty(x) = 0$ //

Second, check that $\lim\limits_{j \to \infty} \|f_j - f_\infty\|_1 = \frac{1}{2}$

as we prove,

$f_\infty = 0$, so

$$\|f_j - f_\infty\|_1$$

$$= \int_0^1 |f_j|\,dx \quad (\because x \in [0,1])$$

$$= \underbrace{\int_0^{\frac{1}{2j}} |f_j|\,dx + \int_{\frac{1}{2j}}^{\frac{1}{j}} |f_j|\,dx}\, + \int_{\frac{1}{j}}^1 |f_j|\,dx$$

$$= \left(\frac{1}{2j} \times j \times \frac{1}{2}\right) \times 2 + 0$$

$$= \frac{1}{2}$$