Bijective Relations Between Spectral Measures, Spectral families, Self-adjoint operators, and Strongly Continuous Unitary Groups

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This report aims to explore the bijective relations between spectral measures, spectral families, self-adjoint operators, and strongly continuous unitary groups. The spectral theorem and Stone's theorem are fundamental results that establish these relations.

1 Definitions

Before delving into the bijective relations between spectral measures, spectral families, self-adjoint operators, and strongly continuous unitary groups, let us begin by revisiting the definitions of these concepts.

Definition 1: Spectral Family

A spectral family, or a resolution of the identity, is a family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of orthogonal projections in $\mathcal{H}$ satisfying:

(i) The family is non-decreasing, namely one has $E_\lambda E_\mu = E_{\min\{\lambda,\mu\}}$,

(ii) The family is strongly right continuous, namely $E_\lambda = E_{\lambda+0} = s - \lim_{\varepsilon\searrow 0} E_{\lambda+\varepsilon}$,

(iii) $s - \lim_{\lambda\to-\infty} E_\lambda = \mathbf{0}$ and $s - \lim_{\lambda\to\infty} E_\lambda = \mathbf{1}$ ([2], pg. 47).

From this definition, one can observe that if $\mu = \lambda$, one would have the idempotency property ($E_\lambda^2 = E_\lambda$) which is one of the important properties that need to be satisfied by a projection operator. Moreover, λ and μ are real parameters representing points in the spectrum of the self-adjoint operator, and $\mathcal{H}$ denotes the Hilbert space on which the operator acts.

Definition 2: Spectral Measure

A spectral measure is a function $E : \mathcal{A}_B \to P(\mathcal{H})$ satisfying the following properties:

(i) $E(\emptyset) = \mathbf{0}$ and $E(\mathbb{R}) = \mathbf{1}$,

(ii) If $\{V_k\}_{k\in\mathbb{N}}$ is a family of disjoint Borel sets, then $E(\cup_k V_k) = \sum_k E(V_k)$. ([2] pg. 48, [1] pg. 3)

From the definition, the real spectral measure E satisfies two fundamental properties. Firstly, the spectral measure E assigns the zero projection to the empty set $\emptyset$ and the identity projection I (identity operator) to the entire space $\mathbb{R}$. Secondly, the spectral measure E is additive for disjoint Borel sets, meaning that the projection onto the union of disjoint Borel sets is equal to the sum of the projections onto each individual set in the family. Generally, the spectral measure can be complex ([1], pg. 3), however, in the context of self-adjoint operators, the spectral measures are real.

Definition 3: Bounded Self-adjoint Operator

A bounded linear operator $B \in \mathcal{B}(\mathcal{H})$ is called self-adjoint or Hermitian if $B^* = B$, or equivalently if for any $f, g \in \mathcal{H}$ one has ([2], pg. 37)

$$\langle f, Bg\rangle = \langle Bf, g\rangle.$$

The condition $B^* = B$ ensures that the operator B is symmetric with respect to the inner product. Here, $\mathcal{B}(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space $\mathcal{H}$. This property is of great significance as it guarantees that self-adjoint operators have real eigenvalues (which will be proved later in the third section). The concept of self-adjoint operators is fundamental in both spectral theory and quantum mechanics, making this definition a cornerstone in the study of linear operators on Hilbert spaces. Then, we want to consider the extended case, namely the unbounded case for the self-adjoint operator

Definition 4: Adjoint Operator

Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ be a densely defined linear operator on a Hilbert space $\mathcal{H}$. The adjoint operator A^* of A is the operator defined by

$$\mathcal{D}(A^*) := \{f \in \mathcal{H} \,|\, \exists f^* \in H \text{ with } \langle f^*, g\rangle = \langle f, Ag\rangle \text{ for all } g \in \mathcal{D}(A)\},$$

and $A^* f := f^*$ for all $f \in \mathcal{D}(A^*)$ ([2], pg. 42).

Given a densely defined linear operator A on a Hilbert space $\mathcal{H}$, its adjoint operator A^* is introduced. This adjoint is defined on a specific domain $\mathcal{D}(A^*)$, which consists of those vectors f for which there exists another vector f^* in $\mathcal{H}$ satisfying a particular inner product condition involving A and g, where g takes values from the domain $\mathcal{D}(A)$. For the unbounded operator, the operator $(A, \mathcal{D}(A))$ is self-adjoint if the domain $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A^* = A$.

Definition 5: Unitary Operator

An element $U \in \mathcal{B}(\mathcal{H})$ is a unitary operator if one has ([2], pg. 38)

$$UU^* = U^*U = \mathbf{1}.$$

This definition states that an operator $U \in \mathcal{B}(\mathcal{H})$ is considered a unitary operator if it satisfies the conditions $UU^* = U^*U = \mathbf{1}$. It's important to highlight that when $\mathcal{H}$ has infinite dimensions, the conditions $UU^* = 1$ and $U^*U = 1$ do not hold the same meaning, namely they are not equivalent. Unitary operators are essential in quantum mechanics and have properties that preserve the inner product and norm of vectors, making them valuable tools in various mathematical and physical contexts.

2 One-to-one correspondence relations

2.1 Spectral Families and Spectral Measures

The bijective relation between spectral family and spectral measure is described as follows: ([2], pg. 48)

Given a spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ one can firstly define

$$E((a,b]) := E_b - E_a, \qquad a, b \in \mathbb{R}$$

and then, one can extend this definition for all sets $V \in \mathcal{A}_B$. Additionally, a notable outcome of this construction is that one has

$$E\left(\bigcup_k V_k\right) = \sum_k E(V_k)$$

whenever V_k is a countable family of disjoint elements of $\mathcal{A}_B$. This definition ($E((a,b]) := E_b - E_a,\ a, b \in \mathbb{R}$) is precisely the spectral measure. Therefore, for a spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ one can associate with the corresponding spectral measure $E : \mathcal{A}_B \to P(\mathcal{H})$, where $\mathcal{A}_B$ is the set of all Borel sets on $\mathbb{R}$ which is also called the Borel σ-algebra and the map E is projection-valued where $P(\mathcal{H})$ represents the set of projection operators on the Hilbert space $\mathcal{H}$. This spectral measure is bounded from below if there exists a $\lambda_- \in \mathbb{R}$ such that $E_\lambda = \mathbf{0}$ for all $\lambda < \lambda_-$, and it is bounded from above if there exists a $\lambda_+ \in \mathbb{R}$ such that $E_\lambda = \mathbf{1}$ for all $\lambda > \lambda_+$.

Overall, the process of defining the spectral measure E from the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ establishes a direct and unique connection between them. This means that the spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ and the spectral measure E has a one-to-one correspondence.

2.2 Spectral Families and Self-adjoint Operators (Spectral Theorem)

Before we delve into the one-to-one correspondence, let us define the spectral integral given the spectral measure E. For a continuous function $\varphi : [a, b] \to \mathbb{C}$ and any spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$. This integral can be realized using a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ and a corresponding collection of points $\{y_j\}$ with $y_j \in (x_{j-1}, x_j)$ such that we can define the following operator

$$\sum_{j=1}^{n} \varphi(y_j)E((x_{j-1}, x_j]).$$

The expression above yields an operator that strongly converges to an element of the bounded operators on $\mathcal{H}$ as the partitions become finer and such that the following operator can be defined ([2], pg. 48).

$$\int_a^b \varphi(\lambda)\, E(d\lambda).$$

Since we have defined the spectral integral, let us consider the following lemma.

Lemma 1

Let $\varphi : \mathbb{R} \to \mathbb{C}$ be continuous, and let us set

$$\mathcal{D}_\varphi := \left\{ f \in \mathcal{H} \middle| \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 m_f(\mathrm{d}\lambda) < \infty \right\}$$

where the measure $m_f(\mathrm{d}\lambda)$ is defined as $m_f(\mathrm{d}\lambda) := \langle f, E(\mathrm{d}\lambda)f\rangle$. Then, the pair $(\int_{-\infty}^{\infty} \varphi(\lambda)E(\mathrm{d}\lambda), \mathcal{D}_\varphi)$ defines a closed linear operator on $\mathcal{H}$. This operator is self-adjoint if and only if φ is a real function ([2], pg. 49).

From **Lemma 1**, given a real spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, there exists a self-adjoint operator $(\int_{-\infty}^{\infty} \lambda E(\mathrm{d}\lambda), \mathcal{D}_{id})$ and $\mathcal{D}_{id}$ is defined as follows

$$\mathcal{D}_{id} := \left\{ f \in \mathcal{H} \middle| \int_{-\infty}^{\infty} \lambda^2 m_f(\mathrm{d}\lambda) < \infty \right\}$$

This operator is called the self-adjoint operator associated with a spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$.

The converse is stated in the theorem below

Theorem 1: Spectral Theorem

Given a self-adjoint operator $(A, \mathcal{D}(A))$ on a Hilbert space $\mathcal{H}$, there exists a unique spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ associated with $(A, \mathcal{D}(A))$ such that $A = \int_{-\infty}^{\infty} \lambda E(\mathrm{d}\lambda)$ and $\mathcal{D}(A) = \mathcal{D}_{id}$. ([2], pg. 49).

Overall, spectral families and self-adjoint operators are connected through the spectral theorem, which establishes a bijective relation between them. The spectral family captures the essential spectral properties of the self-adjoint operator, providing a powerful tool for understanding the operator's spectrum and its corresponding projections.

2.3 Self-adjoint Operators and Strongly Continuous Unitary Groups (Stone's Theorem)

A strongly continuous unitary group is a one-parameter family of unitary operators on a Hilbert space that varies continuously with respect to a real parameter. From **Definition 3.8.7** and **Proposition 3.8.8** in ([2], pg. 50), one can now proceed to state Stone's theorem

Theorem 2: Stone's Theorem

There exists a bijective correspondence between self-adjoint operators on $\mathcal{H}$ and strongly continuous unitary groups on $\mathcal{H}$. More precisely, if A is a self-adjoint operator on $\mathcal{H}$, then $\{e^{-itA}\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group, while if $\{U_t\}_{t\in\mathbb{R}}$ is a strongly continuous unitary group, one sets

$$\mathcal{D}(A) := \left\{ f \in \mathcal{H} \middle| \exists s - \lim_{t\to 0} \frac{1}{t}[U_t - 1]f \right\}$$

and for $f \in \mathcal{D}(A)$ one sets $Af = s - \lim_{t\to 0} \frac{i}{t}[U_t - 1]f$, and then $(A, \mathcal{D}(A))$ is a self adjoint operator.

These strongly continuous unitary groups are commonly used to describe the time-evolution in quantum mechanics.

2.4 Spectral Families and Strongly Continuous Unitary Groups

The one-to-one correspondence between spectral families and strongly continuous unitary groups can be stated as follows, given a spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ one can associate the spectral family with a self-adjoint operator $(A, \mathcal{D}(A))$ by spectral theorem, then, there exists a unique strongly continuous unitary group $\{U(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}$ defined by $U(t) = e^{-itA}$ for all $t \in \mathbb{R}$ by Stone's theorem. Conversely, for any strongly continuous unitary group $U(t)_{t\in\mathbb{R}}$ on $\mathcal{H}$, there exists a unique self-adjoint operator A on $\mathcal{D}(A)$ such that $U(t) = e^{-itA}$ for all $t \in \mathbb{R}$ and then, one can associate the obtained self-adjoint operator $(A, \mathcal{D}(A))$ with a unique spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ by spectral theorem.

References

[1] Sam Raskin. *Spectral Measures and the Spectral Theorem.* 2006.

[2] Serge Richard. *Special Mathematics Lecture: Introduction to Functional Analysis.* 2023.