On Distributions

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This report will first prove **Theorem 1.1.10** as a prerequisite to discuss the order and support of the three distributions in **Examples 1.1.8**. In this report, we assert (without proof) that the three maps given in **Examples 1.1.8** are indeed distributions.

1 Proof of Theorem 1.1.10

Theorem 1.1.10

A map $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{K}$ belongs to $\mathcal{D}'(\mathbb{R}^n)$ if and only if T is linear and if for any $Y \in \mathbb{R}^n$ and any $r > 0$ there exists $c > 0$ and $m \in \mathbb{N}$ such that

$$|T(f)| \leq c \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_\infty \tag{1.1.7}$$

for all $f \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp}(f) \subset \overline{\mathcal{B}_r(Y)}$.

This statement is an "if and only if" statement, so both the forward and backward statements must be proven.

As a reminder, the definition of a distribution in $\mathbb{R}^n$ is:

Definition 1.1.7 (Distribution on $\mathbb{R}^n$)

A map $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{K}$ is a distribution on $\mathbb{R}^n$ if it satisfies:

1. T is linear.
2. $(f_j)_{j \in \mathbb{N}} \xrightarrow{j \to \infty} f_\infty$ in $\mathcal{D}(\mathbb{R}^n) \Longrightarrow (T(f_j)) \xrightarrow{j \to \infty} T(f_\infty)$ in $\mathbb{K}$

Forward Statement

This proof will be a proof by contradiction.

Suppose that there exists a $Y \in \mathbb{R}^n$ and $r > 0$ such that for any $m \in \mathbb{N} \backslash \{0\}$, $\exists f_m \in \mathcal{D}(\overline{\mathcal{B}_r(Y)})$ such that

$$|T(f_m)| > m \sum_{|\alpha| \leq m} \|\partial^\alpha f_m\|_\infty .$$

Note that if $f_m = 0$, then this inequality is impossible, since in that case, both the left-hand side and right-hand side will be 0, and the equation will be $0 > 0$, which is obviously false. Thus, for at least one $X \in \mathbb{R}^n$,

$$0 < |f_m(X)| \leq \|f_m\|_\infty \leq \sum_{|\alpha| \leq m} \|\partial^a f_m\|_\infty$$

Let us define a function $g_m : \mathcal{D}(\overline{\mathcal{B}_r(Y)}) \to \mathbb{K}$, which satisfies:

$$g_m = \frac{f_m}{m \displaystyle\sum_{|\alpha| \leq m} \|\partial^a f_m\|_\infty}.$$

Observe that g_m is simply f_m divided by a nonzero constant. Since $\mathcal{D}(\overline{\mathcal{B}_r(Y)})$ is a vector space, it is closed under addition and scalar multiplication, so $g_m \in \mathcal{D}(\overline{\mathcal{B}_r(Y)})$. Then, one makes two observations :

The first observation is that for any $X \in \mathbb{R}^n$, one has:

$$|g_m(X)| = \frac{|f_m(X)|}{m \sum_{|\alpha| \le m} \|\partial^a f_m\|_\infty} \le \frac{\|f_m\|_\infty}{m \sum_{|\alpha| \le m} \|\partial^a f_m\|_\infty} \le \frac{\sum_{|\alpha| \le m} \|\partial^a f_m\|_\infty}{m \sum_{|\alpha| \le m} \|\partial^a f_m\|_\infty} = \frac{1}{m}.$$

As $\frac{1}{m} \to 0$ as $m \to \infty$, then $g_m(X)$ goes to 0 for all $X \in \mathbb{R}^n$ as $m \to \infty$. Therefore, one concludes that

$$g_m \xrightarrow{m \to \infty} 0 \text{ in } \mathcal{D}(\mathbb{R}^n).$$

The second observation is that since T is linear, we have $T(\lambda f) = \lambda T(f)$ for any $\lambda \in \mathbb{K}$. As such,

$$|T(g_m)| = \frac{|T(f_m)|}{m \sum_{|\alpha| \le m} \|\partial^a f_m\|_\infty} > 1.$$

The equality to the left is permitted since $m \ge 1$ and the sum is strictly positive, and the inequality to the right is obtained by the construction of $|T(f_m)|$. One observes that for any linear map $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{K}$,

$$T(0) = T(0 + 0) \iff T(0) + T(0) = T(0) \implies T(0) = 0,$$

where $T(0)$ is understood to be the functional T acting on the constant zero function $0 \in \mathcal{D}(\mathbb{R}^n)$. On Putting the results of the two observations together, one has:

$$g_m \xrightarrow{m \to \infty} 0 \text{ but } |T(g_m)| > 1 \neq 0 \ \forall m \in \mathbb{N} \backslash \{0\},$$

which presents a contradiction, completing the proof. □

However, there remains one apparent problems with this proof: what if $m = 0$? In truth, this is not a problem at all. In the case of $m = 0$: let there be a distribution $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{K}$ and $Y \in \mathbb{R}^n$ and $r > 0$, such that some $c > 0$ and $m = 0$ satisfies (1.1.7). In this case, the inequality reads

$$|T(f)| \le c \sum_{|\alpha| \le 0} \|\partial^\alpha f\|_\infty .$$

However, observe that the L^∞ norm $\|\cdot\|_\infty$ of any function is always positive, so one has:

$$|T(f)| \le c \sum_{|\alpha| \le 0} \|\partial^\alpha f\|_\infty \le c \sum_{|\alpha| \le 1} \|\partial^\alpha f\|_\infty .$$

Therefore, for any value of m that satisfies (1.1.7), any number $m' \in \mathbb{N}$ that is greater than m (i.e., $n \ge m$) will also satisfy the inequality.

Backward Statement

Define two new sequences $(g_j)_{j \in \mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ by $g_j = f_j - f_\infty \in \mathcal{D}(\mathbb{R}^n)$ and $(S_j)_{j \in \mathbb{N}} \subset \mathbb{K}$ defined by $S_j = T(f_j) - T(f_\infty) \in \mathbb{K}$. One can then rename g_j to f_j and S_j to $T(f_j)$ to write a statement equivalent to the second condition of **Definition 1.1.7**:

$$(f_j)_{j \in \mathbb{N}} \xrightarrow{j \to \infty} 0 \text{ in } \mathcal{D}(\mathbb{R}^n) \implies (T(f_j)) \xrightarrow{j \to \infty} 0 \text{ in } \mathbb{K}$$

Again, it is important to not confuse the zero function $0 \in \mathcal{D}(\mathbb{R}^n)$ with the number zero $0 \in \mathbb{K}$. The 0 in the antecedent is the zero function, while the 0 in the consequent is the number zero.

Returning to the theorem. By assumption, T is linear, so condition 1 is fulfilled. Let $(f_j)_{j\in\mathbb{N}} \in \mathcal{D}(\mathbb{R}^n)$ be a sequence of functions that converges to 0 in $\mathcal{D}(\mathbb{R}^n)$ as $j \to \infty$, with $\text{supp}(f_j) \subset \overline{\mathcal{B}_r(Y)}$ for the given $Y \in \mathbb{R}^n$ and $r > 0$. Choose $c > 0$ and $m \in \mathbb{N}$ such that it satisfies (1.1.7). Then,

$$0 \leq |T(f_j)| \leq c \sum_{|\alpha|\leq m} \|\partial^\alpha f_j\|_\infty$$

Taking the limit as $j \to \infty$, and observing that $\|\partial^\alpha f_j\|_\infty \xrightarrow{j\to\infty} \|\partial^\alpha 0\|_\infty = 0$, one has:

$$0 \leq \lim_{j\to\infty} |T(f_j)| \leq c \sum_{|\alpha|\leq m} \lim_{j\to\infty} \|\partial^\alpha f_j\|_\infty = 0 \Longrightarrow \lim_{j\to\infty} |T(f_j)| = 0$$

As such, the second condition of **Definition 1.1.7** is fulfilled, and one concludes that $T \in \mathcal{D}'(\mathbb{R}^n)$. □

2 Order and Support of a Distribution

In this section, to illustrate the concepts of order and support of distributions more clearly, three examples of distributions in **Examples 1.1.8** are used, namely:

1. The regular distribution T_h defined by

$$T_h(f) := \int_{\mathbb{R}^n} h(X)f(X)\,\mathrm{d}X,$$

   where h is a locally integrable function on $\mathbb{R}^n$ (i.e. $h \in L^1_{loc}(\mathbb{R}^n)$).

2. The Dirac delta distribution for any $Y \in \mathbb{R}^n$, denoted by δ_Y, defined by

$$\delta_Y(f) = f(Y).$$

3. The α-derivative of the Dirac delta distribution for any $Y \in \mathbb{R}^n$, denoted by δ^α_Y, defined by

$$\delta^\alpha_Y(f) := (-1)^{|\alpha|}\left[\partial^\alpha f\right](Y).$$

The proofs that these maps are indeed distributions are not included in this document.

2.1 Order of Distribution

In the framework of **Theorem 1.1.10**, the smallest possible value for m such that m can be made independent from Y and r is set to be the order of the distribution T. To illustrate this concept, the orders of the three distributions above are to be determined.

1. T_h: $\forall\, Y \in \mathbb{R}^n$, $r > 0$ and f satisfying the conditions of **Theorem 1.1.10**, set $K := \overline{\mathcal{B}_r(Y)}$. one has:

$$|T_h(f)| = \int_K |h(X)f(X)|\;\mathrm{d}X = \int_K |h(X)|\,|f(X)|\;\mathrm{d}X.$$

   By definition, $\|f\|_\infty = \sup_{X\in K} |f(X)|$, so:

$$|T_h(f)| \leq \int_K |h(X)|\;\mathrm{d}X \cdot \|f\|_\infty.$$

   Since $h \in L^1_{loc}(\mathbb{R}^n)$, the integral will simply evaluate to a constant. Setting this to be c, one obtains:

$$|T_h(f)| \leq c\,\|f\|_\infty = c \sum_{|\alpha|\leq 0} \|f\|_\infty.$$

   Therefore, the distribution T_h is of order 0.

2. δ_Y: for any $Y \in \mathbb{R}^n$, one has:

$$|\delta_Y(f)| = |f(Y)| \leq \|f\|_\infty = 1 \sum_{|\alpha| \leq 0} \|f\|_\infty .$$

   As such, the Dirac delta distribution δ_Y is of order 0.

3. δ_Y^α: for any $Y \in \mathbb{R}^n$, one has:

$$|\delta_Y^\alpha(f)| = |[\partial^\alpha f](Y)| \leq \|\partial^\alpha f\|_\infty \leq 1 \sum_{|\beta| \leq |\alpha|} \left\|\partial^\beta f\right\|_\infty .$$

   As such, the distribution δ_Y^α is of order $|\alpha|$.

2.2 Support of Distribution

In contrast to the support of functions, which are defined as the subset of their domain where they take a nonzero value, support of distributions are instead found by first looking at the subsets of their domain where they vanish. To understand the concept of the support of a distribution well, one must first study the notion of vanishing distributions.

Definition (Support of a Distribution)

A distribution $T : U \to \mathbb{K}$ is said to vanish in an open subset V of U if $T(f)$ for all $f \in \mathcal{D}(U)$ where $\mathrm{supp}(f) \subseteq V$. The union of all such open subsets is an open set. The support of a distribution is defined to be the complement of said union of open subsets. By this definition, the support of a distribution is always a closed subset, similar to the support of functions. Mathematically, it is written as:

$$\mathrm{supp}(T) = U \backslash \bigcup \{V \mid \forall\ f \mid \mathrm{supp}(f) \subseteq V \Longrightarrow T(f) = 0\}$$

Now, the support of the three distributions in Example 1.1.8 are to be found.

1. T_h: From the expression of $T_h(f)$, with $\mathrm{supp}(f) = \overline{\mathcal{B}_r(Y)} = K$ arbitrary,

$$T_h(f) = \int_K h(X) f(X)\, \mathrm{d}X,$$

   one observes that for this expression to vanish, either $h(X) = 0$ or $f(X) = 0$. Since $f(X) = 0$ depends solely on the test function, that condition can not be used to define the support of the distribution. As such, $T_h(f)$ will vanish if $K \in \mathbb{R}^n \backslash \mathrm{supp}(h)$. Then, by the definition of the support of a distribution:

$$\mathrm{supp}(T_h) = \mathbb{R}^n \backslash \{\mathbb{R}^n \backslash \mathrm{supp}(h)\} = \mathrm{supp}(h).$$

   Therefore, the support of the distribution T_h is simply the support of the function h.

2. δ_Y: Observe that for arbitrary $f \in \mathcal{D}(\mathbb{R}^n)$ such that $Y \notin \mathrm{supp}(f)$, one has:

$$\delta_Y(f) = f(Y) = 0.$$

   The union of all such open subsets is:

$$\bigcup_{f \in \mathcal{D}(\mathbb{R}^n)} \{\mathrm{supp}(f) \mid Y \notin \mathrm{supp}(f)\} = \mathbb{R}^n \backslash \{Y\}.$$

   Therefore, by the definition of the support of a distribution,

$$\mathrm{supp}(\delta_Y) = \mathbb{R}^n \backslash \{\mathbb{R}^n \backslash \{Y\}\} = \{Y\}.$$

   The support of the Dirac delta distribution around point Y is simply the point Y itself.

3. δ_Y^α: very similarly to δ_Y, observe that for arbitrary $f \in \mathcal{D}(\mathbb{R}^n)$ such that $Y \notin \text{supp}(f)$, we have that $Y \notin \text{supp}(\partial^\alpha f)$, so for these f, $\delta_Y^\alpha(f) = 0$. The union of all such open subsets are exactly the same as the case of δ_Y, namely:
$$\bigcup_{f \in \mathcal{D}(\mathbb{R}^n)} \{\text{supp}(f) \mid Y \notin \text{supp}(f)\} = \mathbb{R}^n \backslash \{Y\}.$$
And as such, the support of δ_Y^α is exactly the same as δ_Y: the point Y.