Exercise 1.1.9 (Nagoya University, G30 program)

Finally, show that $\partial^\alpha \delta_r$ belongs to $D'(\mathbb{R}^n)$

From $\partial^\alpha \delta_r(f) := (-1)^{|\alpha|}[\partial^\alpha f](Y)$

1 $\partial^\alpha\delta_r(f+\lambda g) = (-1)^{|\alpha|}[\partial^\alpha(f+\lambda g)](Y)$

$= (-1)^{|\alpha|}[\partial^\alpha f + \partial^\alpha \lambda g](Y)$

$= (-1)^{|\alpha|}[\partial^\alpha f](Y) + (-1)^{|\alpha|}[\partial^\alpha \lambda g](Y)$

$= \partial^\alpha \delta_r(f) + \lambda \partial^\alpha \delta_r(g)$ (Then f, g are functions)

▨

2 Then assume that $f_j \to f$ when $j \to \infty$

Fix $\varepsilon > 0$ and $\alpha \in \mathbb{N}^n$

$\exists N \in \mathbb{N}$ s.t. for $j \geq N$

$\|\partial^\alpha f_j - \partial^\alpha f_\infty\|_\infty = \sup_{x \in \mathbb{R}^n} |\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)| \leq \varepsilon$

Then $|\partial^\alpha \delta_r(f_j) - \partial^\alpha \delta_r(f_\infty)|$

$= |(-1)^{|\alpha|}(\delta_r(\partial^\alpha f_j) - \delta_r(\partial^\alpha f_\infty))|$

$= |\partial^\alpha f_j(Y) - \partial^\alpha f_\infty(Y)|$

$\leq \sup_{x \in \mathbb{R}^n} |\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)|$

$= \|\partial^\alpha f_j - \partial^\alpha f_\infty\|_\infty$

$\leq \varepsilon$ for $j \geq N$

Then $\lim_{j \to \infty} \partial^\alpha \delta_r(f_j) = \partial^\alpha \delta_r(f_\infty)$, as required

▨

Katayama Marin / 022300143