Introduction to Functional Analysis - Proofs of Some Useful Inequalities Valid in Hilbert Spaces

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June 2023

Introduction

The following proofs have been inspired by the methods laid out in the textbook *Hilbert Space Methods in Quantum Mechanics*[1]. Below are the useful inequalities for elements of Hilbert spaces which we wish to prove. Let $\mathcal{H}$ be an arbitrary Hilbert Space. For any $f, g \in \mathcal{H}$ the following inequalities hold:

$$|\langle f,g\rangle| \leq \|f\|\ \|g\| \qquad \text{Schwarz inequality} \quad (1)$$

$$\|f+g\| \leq \|f\| + \|g\| \qquad \text{Triangle inequality} \quad (2)$$

$$\|f+g\|^2 \leq 2\|f\|^2 + 2\|g\|^2 \quad (3)$$

$$\big|\, \|f\| - \|g\| \,\big| \leq \|f-g\| \quad (4)$$

I will also use the following properties for elements of a Hilbert space.

> The following is an excerpt from *Hilbert Space Methods in Quantum Mechanics*[1].
>
> With each couple $\{f, g\}$ of elements of $\mathcal{H}$ there is associated a complex number $\langle f, g\rangle$, and this association has the following properties:
>
> $$\langle g,f\rangle = \overline{\langle f,g\rangle} \qquad \forall f,g \in \mathcal{H} \quad (5)$$
>
> $$\langle f, g+\alpha h\rangle = \langle f,g\rangle + \alpha\langle f,h\rangle \qquad \forall \alpha \in \mathbb{C}, \forall f,g,h \in \mathcal{H} \quad (6)$$
>
> $$\langle f,f\rangle > 0 \qquad \text{except for } f = 0. \quad (7)$$
>
> One then defines
>
> $$\|f\| := [\langle f,f\rangle]^{\frac{1}{2}}. \quad (8)$$

Proof of $|\langle f,g\rangle| \leq \|f\|\ \|g\|$

Special Case of $f = g$

Consider:

$$\begin{aligned}
|\langle f,g\rangle| &= |\langle f,f\rangle| \\
&= \|f\|^2 && \text{(by definition of the inner product.}^{8}\text{)} \\
&= \|f\|\ \|f\| \\
|\langle f,g\rangle| &\leq \|f\|\ \|g\| && \text{when } f = g.
\end{aligned}$$

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General Case of $\forall f, g \in \mathcal{H}$

Consider for any $\alpha \in \mathbb{C}$:

$$\begin{aligned}
0 \leq \|f+\alpha g\|^2 &= \langle f+\alpha g, f+\alpha g\rangle && \text{(by 8.)} \\
&= \langle f+\alpha g, f\rangle + \alpha\langle f+\alpha g, g\rangle && \text{(by 6.)} \\
&= \overline{\langle f, f+\alpha g\rangle} + \alpha\overline{\langle g, f+\alpha g\rangle} && \text{(by 5.)} \\
&= \overline{\langle f,f\rangle + \alpha\langle f,g\rangle} + \alpha\,\overline{(\langle g,f\rangle + \alpha\langle g,g\rangle)} && \text{(by 6.)} \\
&= \langle f,f\rangle + \overline{\alpha}\langle g,f\rangle + \alpha\langle f,g\rangle + \alpha\overline{\alpha}\langle g,g\rangle && \text{(by 5 and linearity of the complement.)}
\end{aligned}$$

Hence one gets

$$\|f + \alpha g\|^2 = \|f\|^2 + \overline{\alpha}\langle g, f\rangle + \alpha\langle f, g\rangle + |\alpha|^2\|g\|^2 . \tag{9}$$

Let $\alpha = -\frac{\langle g,f\rangle}{\|g\|^2}$ so the inequality becomes

$$\begin{aligned} 0 &\leq \|f\|^2 + \overline{-\frac{\langle g, f\rangle}{\|g\|^2}}\langle g, f\rangle + -\frac{\langle g, f\rangle}{\|g\|^2}\langle f, g\rangle + \left|-\frac{\langle g, f\rangle}{\|g\|^2}\right|^2\|g\|^2 \\ &\leq \|f\|^2 - \frac{\langle f, g\rangle\overline{\langle f, g\rangle}}{\|g\|^2} - \frac{\langle f, g\rangle\overline{\langle f, g\rangle}}{\|g\|^2} + \frac{\langle f, g\rangle\overline{\langle f, g\rangle}}{\|g\|^2} \\ &\leq \|f\|^2 - \frac{|\langle f, g\rangle|^2}{\|g\|^2} \\ |\langle f, g\rangle| &\leq \|f\|\ \|g\| . \end{aligned}$$

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Proof of $\|f + g\| \leq \|f\| + \|g\|$

During the proof above we obtained the result

$$\|f + \alpha g\|^2 = \|f\|^2 + \overline{\alpha}\langle g, f\rangle + \alpha\langle f, g\rangle + |\alpha|^2\|g\|^2 \qquad \forall f, g \in \mathcal{H}$$

Starting from this and letting $\alpha = 1$ we infer that

$$\begin{aligned} \|f + g\|^2 &= \|f\|^2 + \langle g, f\rangle + \langle f, g\rangle + \|g\|^2 \\ &\leq \|f\|^2 + |\langle g, f\rangle| + |\langle f, g\rangle| + \|g\|^2 \\ &\leq \|f\|^2 + \|g\|\ \|f\| + \|f\|\ \|g\| + \|g\|^2 && \text{(using the Schwartz inequality.[1])} \\ &\leq (\|f\| + \|g\|)^2 \\ \|f + g\| &\leq \|f\| + \|g\| && \forall f, g \in \mathcal{H} \end{aligned}$$

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Proof of $\|f + g\|^2 \leq 2\|f\|^2 + 2\|g\|^2$

Again using (9) with $\alpha = -1$

$$0 \leq \|f - g\|^2 = \|f\|^2 - \langle g, f\rangle - \langle f, g\rangle + \|g\|^2 . \qquad \forall f, g \in \mathcal{H}$$

We can see that

$$\langle g, f\rangle + \langle f, g\rangle \leq \|f\|^2 + \|g\|^2 .$$

Then substituting into (9) but with $\alpha = 1$ we see that

$$\begin{aligned} \|f + g\|^2 &= \|f\|^2 + \langle g, f\rangle + \langle f, g\rangle + \|g\|^2 \\ \|f + g\|^2 &\leq \|f\|^2 + \|f\|^2 + \|g\|^2 + \|g\|^2 \\ \|f + g\|^2 &\leq 2\|f\|^2 + 2\|g\|^2 . \end{aligned}$$

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Proof of $\big|\,\|f\| - \|g\|\,\big| \leq \|f - g\|$

Start by considering $\big|\,\|f\| - \|g\|\,\big|$ for any $f, g \in \mathcal{H}$. Choose the larger of the two to be f_L and the smaller of the two to be f_S such that

$$\big|\,\|f\| - \|g\|\,\big| = \|f_L\| - \|f_S\|$$

Then using (2), which we already proved, we get that

$$\begin{aligned}\|f_L\| - \|f_S\| &= \|f_L - f_S + f_S\| - \|f_S\| \\ &\leq (\|f_L - f_S\| + \|f_S\|) - \|f_S\| \\ &\leq \|f_L - f_S\|\end{aligned}$$

Additionally, it is easy to check that $\|f_L - f_S\| = \|f_S - f_L\|$. Therefore,

$$\big|\, \|f\| - \|g\| \,\big| \leq \|f - g\|\,.$$

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References

1. W. O. Amrein, *Hilbert Space Methods in Quantum Mechanics*, eng (EPFL Press [u.a.], Lausanne, 1. ed, 2009), ISBN: 9781420066814.