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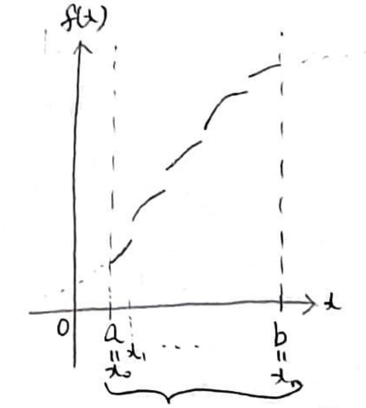
Exercise 2.1.6 Monotone functions are Riemann integrable



P_n: regular partition of $[a,b]$ with n subintervals

1. $U(f,P_n) = \sum_{j=1}^{n} \left(\sup_{x \in [x_{j-1}, x_j]} f(x) \right)(x_j - x_{j-1})$

$= \underline{\sum_{j=1}^{n} f(x_j)(x_j - x_{j-1})}$ //

($\because$ f is increasing in $x \in [a,b]$)

$L(f,P_n) = \sum_{j=1}^{n} \left(\inf_{x \in [x_{j-1}, x_j]} f(x) \right)(x_j - x_{j-1})$

$= \underline{\sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1})}$ //

($\because$ f is increasing in $x \in [a,b]$)

2. $U(f,P_n) - L(f,P_n) = \sum_{j=1}^{n} \{ f(x_j) - f(x_{j-1}) \}(x_j - x_{j-1})$

$= \frac{b-a}{n} \sum_{j=1}^{n} \{ f(x_j) - f(x_{j-1}) \}$

$= \underline{\frac{b-a}{n} \{ f(b) - f(a) \}}$ //

(P_n is the regular partition of $[a,b]$ with n subintervals, so $x_j - x_{j-1} = \frac{b-a}{n}$ for any $j \in \{1, 2, \cdots, n\}$)

3. Since $f \in \mathcal{L}^\infty([a,b])$, both $b-a$ and $f(b)-f(a)$ are finite, for any $\varepsilon > 0$, we set

$$n = \frac{2(b-a)\{f(b)-f(a)\}}{\varepsilon}$$

then $U(f,P_n) - L(f,P_n) = \frac{\varepsilon}{2} \leq \varepsilon$

Therefore, f is Riemann integrable. □