SML Spring 2023 Introduction to functional analysis

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Exercise 1.2.5. [p.7 in the cumulative notes] Check that $Pv\frac{1}{x}$ defined in (1.2.1) is a distribution, and compute the exact expression for $O(\epsilon\ |ln(|\epsilon|)|)$.

Reminder:

Definition 0.1 (The principal value distribution defined on $f \in \mathcal{D}(\mathbb{R})$, p.7 in the cumulative notes)**.**

$$Pv\frac{1}{x}(f) := \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{1}{x} f(x)\, dx = \lim_{\epsilon \searrow 0} \left(\int_{-\infty}^{-\epsilon} \frac{1}{x} f(x)\, dx + \int_{\epsilon}^{\infty} \frac{1}{x} f(x)\, dx \right). \quad (1.2.1)$$

Then, let us observe that

$$\begin{aligned} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{1}{x} f(x)\, dx &= \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \ln(|x|)' f(x)\, dx \\ &= -\lim_{\epsilon \searrow 0} \left(\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \ln(|x|) f(x)'\, dx + O\Big(\epsilon \Big|\ln(|\epsilon|)\Big|\Big) \right) \qquad (1) \\ &= -\int_{\mathbb{R}} \ln(|x|) f'(x)\, dx. \end{aligned}$$

Definition 0.2 (Definition 1.1.6; Convergence in $\mathcal{D}(\mathbb{R}^n)$)**.** $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to $f_\infty \in \mathcal{D}(\mathbb{R}^n)$ if the following conditions are satisfied:

1. For any $\alpha \in \mathbb{N}^n$ one has

$$\sup_{x\in\mathbb{R}} \left|\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)\right| \to 0 \text{ as } j \to \infty,$$

2. There exists $r \in \mathbb{R}$ large enough such that $\mathrm{supp}(f_j) \subset B_r(0)$ for all $j \in \mathbb{N}$.

Definition 0.3 (Definition 1.1.7; Distribution on $\mathcal{D}(\mathbb{R}^n)$)**.** A distribution on $\mathbb{R}^n$ is a continuous linear function on $\mathcal{D}(\mathbb{R}^n)$ with values in $\mathbb{K}$. More precisely, a map $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{K}$ is a distribution on $\mathbb{R}^n$ if it satisfies:

1. $T(f_1 + \lambda f_2) = T(f_1) + \lambda T(f_2)$ for any $f_1, f_2 \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda \in \mathbb{K}$,

2. Whenever the sequence $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ converges to $f_\infty \in \mathcal{D}(\mathbb{R}^n)$, then the sequence $\left(T(f_j)\right)_{j\in\mathbb{N}}$ converges to $T(f_\infty)$ in $\mathbb{K}$ as $j \to \infty$.

Example 0.4 (Example 1.1.8. (1))**.** For any $h \in L^1_{loc}(\mathbb{R}^n)$ we define $T_h \in \mathcal{D}'(\mathbb{R}^n)$ acting on $f \in \mathcal{D}(\mathbb{R}^n)$ as

$$T_h(f) := \int_{\mathbb{R}^n} h(X) f(X)\, dX$$

and call T_h a regular distribution. Thus, any element of $L^1_{loc}(\mathbb{R}^n)$ can be identified with a distribution through the map $h \to T_h$.

Definition 0.5 (Definition 1.2.1; Differentiation of a distribution)**.** For any $T \in \mathcal{D}'(\mathbb{R}^n)$ and for any $\alpha \in \mathbb{N}^n$, we define $\partial^\alpha T$ acting on $f \in \mathcal{D}(\mathbb{R}^n)$ as

$$[\partial^\alpha T](f) := (-1)^{|\alpha|} T(\partial^\alpha f).$$

Answer: I show that $Pv\frac{1}{x}$ is a continuous linear function on $\mathcal{D}(\mathbb{R})$ with values in $\mathbb{K}$ according to Definition 1.1.7.

1. $Pv\frac{1}{x}$ is a linear function on $\mathcal{D}(\mathbb{R})$ with values in $\mathbb{K}$, because the following computation holds from the linearity of integrals.

$$\begin{aligned} Pv\frac{1}{x}(f_1 + \lambda f_2) &= \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{1}{x}(f_1 + \lambda f_2)(x)\,dx \\ &= \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{1}{x} f_1(x)\,dx + \lambda \lim_{\epsilon \searrow 0} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{1}{x} f_2(x)\,dx \\ &= Pv\frac{1}{x}(f_1) + \lambda Pv\frac{1}{x}(f_2) \end{aligned}$$

for any $f_1, f_2 \in \mathcal{D}(\mathbb{R})$ and $\lambda \in \mathbb{K}$.

2. I show that whenever the sequence $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R})$ converges to $f_\infty \in \mathcal{D}(\mathbb{R})$, the sequence $\left(Pv\frac{1}{x}(f_j)\right)_{j\in\mathbb{N}}$ converges to $Pv\frac{1}{x}(f_\infty)$ in $\mathbb{R}$ as $j \to \infty$ according to Definition 1.1.6, Example 1.1.8 (1) and Definition 1.2.1.

Assume that $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R})$ converges to $f_\infty \in \mathcal{D}(\mathbb{R})$. That is,

(i) For any $\alpha \in \mathbb{N}$ one has $\sup_{x\in\mathbb{R}} \left|\partial^\alpha f_j(x) - \partial^\alpha f_\infty(x)\right| \to 0$ as $j \to \infty$,

(ii) There exists $r \in \mathbb{R}$ large enough such that $\mathrm{supp}(f_j) \subset B_r(0)$ for all $j \in \mathbb{N}$.

Then, since

$$Pv\frac{1}{x}(f) = -\int_{\mathbb{R}} \ln(|x|)\, f'(x)\,dx = [\partial T_{\ln(|\cdot|)}](f),$$

one has

$$\begin{aligned} &\left|Pv\frac{1}{x}(f_j) - Pv\frac{1}{x}(f_\infty)\right| \\ &= \left|\int_{\mathbb{R}} \ln(|x|)\, f_j'(x)\,dx - \int_{\mathbb{R}} \ln(|x|)\, f_\infty'(x)\,dx\right| \\ &= \left|\int_{\mathbb{R}} \ln(|x|)\left(f_j'(x) - f_\infty'(x)\right)\,dx\right| \\ &\leq \int_{\mathbb{R}} \left|\ln(|x|)\right| \left|f_j'(x) - f_\infty'(x)\right|\,dx. \end{aligned}$$

Since $(f_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R})$ converges to $f_\infty \in \mathcal{D}(\mathbb{R})$, $\sup_{x\in\mathbb{R}} \left|f_j'(x) - f_\infty'(x)\right| \to 0$ as $j \to \infty$ from the condition (i). And $\left|\ln(|x|)\right| \left|f_j'(x) - f_\infty'(x)\right| = 0$ for $x \in \mathbb{R} \setminus B_r(0)$ from the condition (ii). Thus,

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\ln(|x|)\right| \left|f_j'(x) - f_\infty'(x)\right| \, dx \\
&\le \sup_{y\in B_r(0)} \left|f_j'(y) - f_\infty'(y)\right| \int_{B_r(0)} \ln(|x|) \; dx \to 0 \text{ as } j \to \infty.
\end{aligned}$$

[1] Hence, the sequence $\left(Pv\frac{1}{x}(f_j)\right)_{j\in\mathbb{N}}$ converges to $Pv\frac{1}{x}(f_\infty)$ in $\mathbb{K}$.

Next, I compute the exact expression for $O(\epsilon\left|ln(|\epsilon|)\right|)$ in (1).
Reminder; Big O notation
Let f, the function to be estimated, be a real or complex valued function and let g, the comparison function, be a real valued function. Let both functions be defined on some unbounded subset of the positive real numbers, and $g(x)$ be strictly positive for all large enough values of x. One writes $f(x) = O(g(x))$ if there exists a positive real number M and a real number x_0 such that

$$|f(x)| \le Mg(x) \quad \text{for all } x \ge x_0.$$

[Big O notation in Wikipedia]

Checking the computation of $Pv\frac{1}{x}(f)$ with $\epsilon > 0$ and $L > 0$,

$$\begin{aligned}
Pv\frac{1}{x}(f) &= \lim_{\epsilon\searrow 0} \int_{\mathbb{R}\setminus(-\epsilon,\epsilon)} \frac{1}{x} f(x)\, dx \\
&= \lim_{\epsilon\searrow 0,\ L\nearrow\infty} \left[\int_{-L}^{-\epsilon} \ln(|x|)' f(x)\, dx\right] + \lim_{\epsilon\searrow 0,\ L\nearrow\infty} \left[\int_{\epsilon}^{L} \ln(|x|)' f(x)\, dx\right] \\
&= \lim_{\epsilon\searrow 0,\ L\nearrow\infty} \left[\ln(|x|)f(x)|_{-L}^{-\epsilon} - \int_{-L}^{-\epsilon} \ln(|x|)f(x)'\, dx\right] \\
&\quad + \lim_{\epsilon\searrow 0,\ L\nearrow\infty} \left[\ln(|x|)f(x)|_{\epsilon}^{L} - \int_{\epsilon}^{L} \ln(|x|)f(x)'\, dx\right] \\
&= \lim_{\epsilon\searrow 0,\ L\nearrow\infty} \left[\ln(\epsilon)f(-\epsilon) - \ln(L)f(-L) - \int_{-L}^{-\epsilon} \ln(|x|)f(x)'\, dx\right] \\
&\quad + \lim_{\epsilon\searrow 0,\ L\nearrow\infty} \left[\ln(L)f(L) - \ln(\epsilon)f(\epsilon) - \int_{\epsilon}^{L} \ln(|x|)f(x)'\, dx\right] \\
&= \lim_{\epsilon\searrow 0,\ L\nearrow\infty} \left[-\int_{-L}^{-\epsilon} \ln(|x|)f(x)'\, dx - \int_{\epsilon}^{L} \ln(|x|)f(x)'\, dx\right] \\
&\quad + \lim_{\epsilon\searrow 0,\ L\nearrow\infty} \left[\ln(\epsilon)f(-\epsilon) - \ln(L)f(-L) + \ln(L)f(L) - \ln(\epsilon)f(\epsilon)\right]
\end{aligned} \tag{2}$$

[1] $\ln(|x|) \in L^1_{loc}(\mathbb{R})$. See "On the functions $\frac{1}{x}$ and $\ln(x)$" in Student's reports.

In the expression (2),

$$\begin{aligned}
&\lim_{\epsilon\searrow 0,\ L\nearrow\infty}\Big\{-\int_{-L}^{-\epsilon}\ln(|x|)f(x)'\,dx-\int_{\epsilon}^{L}\ln(|x|)f(x)'\,dx\Big\}\\
&=-\lim_{\epsilon\searrow 0}\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\ln(|x|)f(x)'\,dx.
\end{aligned}$$

And,

$$\begin{aligned}
&\lim_{\epsilon\searrow 0,\ L\nearrow\infty}\Big\{\ln(\epsilon)f(-\epsilon)-\ln(L)f(-L)+\ln(L)f(L)-\ln(\epsilon)f(\epsilon)\Big\}\\
&=\lim_{\epsilon\searrow 0}\ln(\epsilon)\Big\{f(-\epsilon)-f(\epsilon)\Big\}-\lim_{L\nearrow\infty}\ln(L)\Big\{f(-L)-f(L)\Big\}\\
&=\lim_{\epsilon\searrow 0}\ln(\epsilon)\Big\{f(-\epsilon)-f(\epsilon)\Big\}+\lim_{L\nearrow\infty}\ln(L^{-1})\Big\{f(-L)-f(L)\Big\}. \qquad (3)
\end{aligned}$$

Since f is smooth on $\mathbb{R}$, there exists $M>0$ such that

$$\lim_{\epsilon\searrow 0}\frac{|f(-\epsilon)-f(0)|}{\epsilon}\le M \quad\text{and}\quad \lim_{\epsilon\searrow 0}\frac{|f(\epsilon)-f(0)|}{\epsilon}\le M.$$

Thus,

$$\begin{aligned}
&\lim_{\epsilon\searrow 0}\Big|\ln(\epsilon)\left\{f(-\epsilon)-f(\epsilon)\right\}\Big|\\
&\le\lim_{\epsilon\searrow 0}\Big|\ln(\epsilon)\left\{f(-\epsilon)-f(0)\right\}\Big|+\lim_{\epsilon\searrow 0}\Big|\ln(\epsilon)\left\{f(-\epsilon)-f(\epsilon)\right\}\Big|\\
&\le 2M\epsilon\,|\ln(\epsilon)|\,. \qquad (4)
\end{aligned}$$

Also, $f(-L)-f(L)=0$ for L large enough since $f\in\mathcal{D}(\mathbb{R})$. Thus,

$$\lim_{L\nearrow\infty}\ln(L^{-1})\Big\{f(-L)-f(L)\Big\}=0. \qquad (5)$$

Hence, from (4) and (5), the expression (3) is evaluated as

$$\lim_{\epsilon\searrow 0}\left|\ln(\epsilon)\Big\{f(-\epsilon)-f(\epsilon)\Big\}\right|+\lim_{L\nearrow\infty}\left|\ln(L^{-1})\Big\{f(-L)-f(L)\Big\}\right|\le 2M\epsilon\,|\ln(\epsilon)|\,.$$

This is the exact expression for $O\Big(\epsilon\Big|\ln(|\epsilon|)\Big|\Big)$ in the equation (1).

Note that $\lim_{\epsilon\searrow 0}O\Big(\epsilon\Big|\ln(|\epsilon|)\Big|\Big)=0$.
Indeed, by the rule of de L'Hospital,

$$\lim_{\epsilon\searrow 0}\epsilon\Big|\ln(|\epsilon|)\Big|=\lim_{\epsilon\searrow 0}\frac{\Big|\ln(|\epsilon|)\Big|}{1/\epsilon}=\lim_{\epsilon\searrow 0}\frac{\Big|\ln(|\epsilon|)\Big|'}{(1/\epsilon)'}=\lim_{\epsilon\searrow 0}\frac{1/\epsilon}{-1/\epsilon^2}=\lim_{\epsilon\searrow 0}(-\epsilon)=0.$$