SML Spring 2023 Introduction to functional analysis

May 4, 2023 Koichi Kato 522301063

Question and Answer

Reminder:

Definition 0.1 (The set of locally integrable functions on $\mathbb{R}^n$; $L^1_{loc}(\mathbb{R}^n)$).

$$L^1_{loc}(\mathbb{R}^n) := \{h : \mathbb{R}^n \to \mathbb{K} \mid \int_{B_r(Y)} |h(X)|\, dX < \infty \text{ for any } r > 0, Y \in \mathbb{R}^n\}.$$

Question: Do the functions $\frac{1}{x}$ and $ln(|x|)$ on $\mathbb{R}$ belong to $L^1_{loc}(\mathbb{R})$?

Answer [Professor]: Going to infinity at one point does not mean anything for $L^1_{loc}(\mathbb{R})$. For example, the function $1/x$ is not in $L^1_{loc}(\mathbb{R})$. Indeed, take the improper Riemann integral

$$\lim_{\epsilon \searrow 0} \int_\epsilon^1 \frac{1}{x}\, dx = \lim_{\epsilon \searrow 0} \ln(x)|_\epsilon^1 = -\lim_{\epsilon \searrow 0} \ln(\epsilon) = \infty.$$

On the other hand, the function $\ln(|x|)$ is in $L^1_{loc}(\mathbb{R})$. Indeed,

$$\lim_{\epsilon \searrow 0} \int_\epsilon^1 \ln(|x|)\, dx = \lim_{\epsilon \searrow 0} (|x| \ln(|x|) - |x|)\,|_\epsilon^1 = -1 - \lim_{\epsilon \searrow 0} (\epsilon \ln(\epsilon) - \epsilon) = -1,$$

where we used the rule of de L'Hospital;

$$\lim_{\epsilon \searrow 0} (\epsilon \ln(\epsilon)) = \frac{\lim_{\epsilon \searrow 0} \ln(\epsilon)'}{\lim_{\epsilon \searrow 0} 1/\epsilon} = \lim_{\epsilon \searrow 0}(-\epsilon) = 0.$$

Thus, $\ln(|x|)$ is locally integrable on $[0, 1]$, and also on $[-1, 0]$ since $\ln(|x|)$ is an even function. Hence, $\ln(|x|)$ is locally integrable on $\mathbb{R}$.