This document will discuss two theorems on linear operators, which are called '*Riesz Lemma*' and '*Neumann Series*'.

I *Riesz Lemma*

In this section, we will see the proof of '*Riesz Lemma*' whose argument is here:

Riesz Lemma

For any $\varphi \in \mathcal{H}$, there exists a unique $g \in \mathcal{H}$ such that for any $f \in \mathcal{H}$

$$\varphi(f) = \langle g, f \rangle$$

In addition, g satisfies $||\varphi||_{\mathcal{H}} = ||g||$.

I will provide the proof of this theorem by three steps.

On the uniqueness

Let's take an element $g' \in \mathcal{H}$ which satisfies $\varphi(f) = \langle g', f \rangle$. As $\varphi(f)$ is equal to $\langle g, f \rangle$, we can get $\langle g, f \rangle = \langle g', f \rangle$. By the property of inner product: linearity in the second argument, $\langle g - g', f \rangle = 0$. You can get $g - g' = 0$ i.e., $g = g'$ since the element $f \in \mathcal{H}$ is arbitrary, which implies that you can take f as $g - g'$.

On the existence of the above g

When you define a set $\mathcal{M} = \{f \in \mathcal{H} | \varphi(f) = 0\}$, $\mathcal{M} \subsetneq \mathcal{H}$ (this means that $\mathcal{M}$ is included in $\mathcal{H}$ but is not equal to $\mathcal{H}$). Note that this set $\mathcal{M}$ is closed. Therefore, we can apply '*Projection Theorem*' to $\mathcal{M}$, and we can conclude that ,for any $h \in \mathcal{H}$ such that $\varphi(h)$ is not equal to 0, we can decompose $h = h_1 + h_2$ with $h_1 \in \mathcal{M}$ and $h_2 \in \mathcal{M}^{\perp}$. Since $\varphi(f - \frac{\varphi(f)}{\varphi(h_2)} h_2) = 0$ ($\because$ the linearity of function φ), which implies $f - \frac{\varphi(f)}{\varphi(h_2)} h_2 \in \mathcal{M}$. As $h_2 \in \mathcal{M}^{\perp}$, $\left\langle h_2, f - \frac{\varphi(f)}{\varphi(h_2)} h_2 \right\rangle = 0$. This equation is equivalent to

$$\langle h_2, f \rangle = \frac{\varphi(f)}{\varphi(h_2)} ||h_2||^2 \iff \varphi(f) = \frac{\varphi(h_2)}{||h_2||^2} \langle h_2, f \rangle$$

When you set $g = \overline{\frac{\varphi(h_2)}{||h_2||^2}} h_2$, you can get the equation: $\varphi(f) = \langle g, f \rangle$.

In conclusion, there is at least one g which satisfies the statement. (Even though we know this g is unique from the above argument)

On the equation $||\varphi||_{\mathcal{H}} = ||g||$

When you take f such that the norm of f is equal to 1, using Schwarz inequality,

$$|\varphi(f)| = | \langle g, f \rangle | \leq ||f|| \, ||g|| = ||g||$$

When you set $f = \frac{g}{||g||}$,

$$||g|| = \frac{\varphi(g)}{||g||} = \varphi(f) \leq \sup \varphi(f) = ||\varphi||$$

Therefore the equation holds.

II *Neumann Series*

Let's begin with the definition of *Neunmann Series*.

Neumann Series

If $\boldsymbol{B} \in \mathcal{B}(\mathcal{H})$ *and* $||\boldsymbol{B}|| < 1$, *then the operator* **1-B** *is invertible in B(H), with*

$$(\textbf{1-B})^{-1} = \sum_{n=0}^{\infty} \boldsymbol{B}^n$$

and with $||(\textbf{1-B})^{-1}|| \leq (1 - ||\boldsymbol{B}||)^{-1}$. *The series converges in the uniform norm of* $\mathcal{B}(\mathcal{H})$.

Before proving this, we will provide a part of solutions of Exercise 3.2.4 in the lecture note.
For any $\boldsymbol{A},\boldsymbol{B} \in \mathcal{B}(\mathcal{H})$, $||\boldsymbol{AB}|| \leq ||\boldsymbol{A}|| \, ||\boldsymbol{B}||$.
proof.
By the definition of the norm: $||C|| = \sup \frac{||Cf||}{||f||}$, we can conclude that
$||C|| \geq \frac{||Cf||}{||f||} \iff ||Cf|| \leq ||C|| \, ||f|| \; (\forall C \in \mathcal{B}(\mathcal{H}))$. From the above arguments, we obtain

$$||\mathbf{AB}f|| \leq ||\mathbf{A}|| \, ||\mathbf{B}f|| \leq ||\mathbf{A}|| \, ||\mathbf{B}|| \, ||f||$$

Extending this argument, we can easily get $||\mathbf{B}^n|| \leq ||\mathbf{B}||^n$.
Now, we will see the proof of *Neumann Series*.
Since $||\mathbf{B}|| < 1$, $\sum_{n=0}^{\infty} ||B||^n$ converges. Also, as we discussed before, $||\mathbf{B}^n|| \leq ||\mathbf{B}||^n$
Combining the above two properties with the following inequation, one has, for any $m \geq n+1$,

$$||\sum_{k=0}^{m} \mathbf{B}^k - \sum_{k=0}^{n} \mathbf{B}^k|| = ||\sum_{k=n+1}^{m} \mathbf{B}^k|| \leq \sum_{k=n+1}^{\infty} ||\mathbf{B}^k|| \leq \sum_{k=n+1}^{\infty} ||\mathbf{B}||^k \longrightarrow 0 \; (n \to \infty)$$

We can say that $\sum_{k=0}^{m} \mathbf{B}^k$ is Cauchy sequence. As we can see in the page 31 on the lecture note, $\mathcal{B}(\mathcal{H})$ is complete so there exists the limit: **B'** of this sequence in $\mathcal{B}(\mathcal{H})$. Using this **B'**,

$$\mathbf{BB'} = \mathbf{B'B} = \sum_{k=0}^{\infty} \mathbf{B}^{k+1} = \sum_{k=0}^{\infty} \mathbf{B}^k - \mathbf{1} = \mathbf{B'} - \mathbf{1}$$

$$\iff (\mathbf{1} - \mathbf{B})\mathbf{B'} = \mathbf{B'}(\mathbf{1} - \mathbf{B}) = \mathbf{1}$$

This equation shows that the inverse of $\mathbf{1} - \mathbf{B}$ is **B'**.
In addition,

$$||(\mathbf{1} - \mathbf{B})^{-1}|| = ||\mathbf{B'}|| = ||\sum_{k=0}^{\infty} \mathbf{B}^k|| \leq \sum_{k=0}^{\infty} ||\mathbf{B}||^k = \frac{1}{1 - ||\mathbf{B}||}.$$