L^p

This report shows some properties and theorems on L^1 and L^p. The main point of this report is to provide the proof of *Hölder's inequality* and *Minkowski's inequality*. Those statements are indicated below:

Hölder's inequality and Minkowski's inequality

Hölder's inequality
Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, and consider $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then the product fg belongs to $L^1(\Omega)$ and the following inequality holds:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Minkowski's inequality
For $p \geq 1$ and for any $f, g \in L^p(\Omega)$ one has

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Hölder's inequality

We were discussing about L^1-spaces in the previous section. In this section, we will see L^p-spaces, more precisely, we will see two theorems and their proofs, named 'Hölder's inequality' and 'Minkowski's inequality'. Both of those theorems show us the strong connection among L^p-spaces ($p \in \mathbb{R}_{\geq 1}$).
Let's begin with 'Hölder's inequality'. The statement of this theorem is below:

Hölder's inequality

Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, and consider $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then the product fg belongs to $L^1(\Omega)$ and the following inequality holds:

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Before going on this proof, we have to accept one fact(*Lemma 2.6.9.* in the lecture note):

$$\textit{for any } a, b \geq 0,\ ab \leq p^{-1}a^p + q^{-1}b^q$$

This inequality is natural if you consider the function $log\, t$ which is convex downward. More accurately, as $p^{-1}, q^{-1} > 0, p^{-1} + q^{-1} = 1$,

$$log(p^{-1}a^p + q^{-1}b^q) \geq p^{-1}log(a^p) + q^{-1}log(b^q) = log(ab) \qquad (\natural)$$

Proof.
If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $fg = 0$ a.e. so we will consider $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$. Set $a = \frac{|f(x)|}{\|f\|_p}$,

$b = \frac{|g(x)|}{||g||_q}$, then we can employ (♮) for this a, b and get

$$\begin{aligned}
\frac{1}{||f||_p||g||_q}\left|\int_\Omega f(x)g(x)\,dx\right| &\leq \int_\Omega \frac{|f(x)|}{||f||_p}\frac{|g(x)|}{||g||_q}\,dx \\
&\leq \int_\Omega \left\{\frac{|f(x)|^p}{p||f||_p^p} + \frac{|g(x)|^q}{q||g||_q^q}\right\}dx \\
&= \frac{1}{p||f||_p^p}\int_\Omega |f(x)|^p\,dx + \frac{1}{q||g||_q^q}\int_\Omega |g(x)|^q\,dx \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

To put it simply,

$$\int_\Omega |f(x)g(x)|\,dx \leq ||f(x)||_p||g(x)||_q$$

The right side of this equation is exactly what we examined in the last section: $L^1(\Omega)$ norm. That's why *Hölder's inequality* holds. □

In the special case: $p = q = 2$, this inequality is called *Schwarz's inequality*.

Schwarz's inequality: $||fg||_1 \leq ||f||_2||g||_2$

Minkowski's inequality

Secondly, let's move froward to next theorem called *Minkowski's inequality* whose statement is here:

Minkowski's inequality

For $p \geq 1$ and for any $f, g \in L^p(\Omega)$ one has

$$||f + g||_p \leq ||f||_p + ||g||_p$$

Same as the former proof, we have to use one fact here for proving this statement:

$$\textit{For any } a, b \geq 0,\ (a + b)^p \leq 2^{p-1}(a^p + b^p) \qquad (\heartsuit)$$

This inequality is true because the function $\frac{(1+t)^p}{1+t^p}$ $(t \geq 0)$ reaches max point(2^{p-1}) when $t = 1$.

Proof.

If $p = 1$, then the statement is equivalent to the triangle inequality which is true in $L^1(\Omega)$ ($\because$ $L^1(\Omega)$ is Banach space). From now on, we assume $p > 1$. When we utilize (♡),

$$|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|)^p \leq 2^{p-1}(|f(x)|^p + |g(x)|^p)$$

and this inequality implies that $f + g \in L^p(\Omega)$. As $p^{-1} + q^{-1} = 1 \implies q = \frac{p}{p-1}$, $|f + g|^{p-1} \in L^q(\Omega)$ ($\because$ *Hölder's inequality*). Therefore,

$$\begin{aligned}
\int_\Omega |f(x) + g(x)|^p\,dx &\leq \int_\Omega |f(x) + g(x)|^{p-1}|f(x)|\,dx + \int_\Omega |f(x) + g(x)|^{p-1}|g(x)|\,dx \\
&\leq \left(\int_\Omega |f(x) + g(x)|^p\,dx\right)^{\frac{1}{q}}\left\{\left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} + \left(\int_\Omega |g(x)|^p\,dx\right)^{\frac{1}{p}}\right\}
\end{aligned}$$

(Remark: the second inequality is from *Hölder's inequality*)
By dividing the previous inequality by $\left(\int_\Omega |f(x)+g(x)|^p\,dx\right)^{\frac{1}{q}}$, one gets

$$\begin{aligned}\left(\int_\Omega |f(x)+g(x)|^p\,dx\right)^{\frac{1}{p}} &= \left(\int_\Omega |f(x)+g(x)|^p\,dx\right)^{1-\frac{1}{q}} \\ &\leq \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} + \left(\int_\Omega |g(x)|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

,which implies the statement of *Minkowski's inequality*. □

In the end of this chapter, we will look at one interesting fact:

Proposition 1

Whenever $|\Omega| < \infty$, if $p_1 < p_2$, then $L^{p_2}(\Omega) \subset L^{p_1}(\Omega)$ and $\|f\|_{p_1} \leq |\Omega|^{\frac{p_2-p_1}{p_1p_2}}\|f\|_{p_2}$

Proof. By applying the *Hölder's inequality* to the function f with $p = \frac{p_2}{p_1} > 1$ and $q = \frac{p_2}{p_2-p_1}$, we get

$$\int_\Omega |f(x)|^{p_1}\,dx \leq \left(\int_\Omega |f(x)|^{p_2}\right)^{\frac{p_1}{p_2}}\left(\int_\Omega dx\right)^{\frac{p_2-p_1}{p_2}} = \|f\|_{p_2}^{p_1}|\Omega|^{\frac{p_2-p_1}{p_2}}.$$

□