Proofs on Some Distributions

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Exercise 1.1.9 & Exercise 1.1.11

Show that $T_h, \delta_Y, \delta_Y^\alpha \in \mathcal{D}'(\mathbb{R}^n)$ and determine the order of each distributions.

Proof:

1. $T_h \in \mathcal{D}'(\mathbb{R}^n)$. It's given that $T_h(f) = \int_{\mathbb{R}^n} h(X)f(X)dX$ for any $h \in L^1_{loc}(\mathbb{R}^n)$ and for $f \in \mathcal{D}(\mathbb{R}^n)$. Let us prove the linearity of T_h. For some $\lambda \in \mathbb{K}$ and $f_1, f_2 \in \mathcal{D}(\mathbb{R}^n)$, one has

$$T_h(f_1 + \lambda f_2) = \int_{\mathbb{R}^n} h(x)(f_1(X) + \lambda f_2(X))dX = \int_{\mathbb{R}^n} h(X)f_1(X)dX + \lambda \int_{\mathbb{R}^n} h(X)f_2(X)dX$$
$$= T_h(f_1) + \lambda T_h(f_2).$$

Therefore, T_h is linear. Let us fix $\varepsilon > 0$ and $\alpha = (0, 0, \ldots, 0)$. By **Definition 1.1.6** about the convergence in $\mathcal{D}(\mathbb{R}^n)$, there exists $N \in \mathbb{N}$ such that for $j \geq N$.

$$||f_j - f_\infty||_\infty \leq \frac{\varepsilon}{\int_{B_r(Y)} |h(X)|dX}$$

Then

$$\begin{aligned}
|T_h(f_j) - T_h(f_\infty)| &= \left| \int_{\mathbb{R}^n} h(X)f_j(X)dX - \int_{\mathbb{R}^n} h(X)f_\infty(X)dX \right| \\
&= \left| \int_{\mathbb{R}^n} h(X)(f_j(X) - f_\infty(X))dX \right| \\
&\leq \int_{B_r(Y)} |h(X)| \left| f_j(X) - f_\infty(X) \right| dX \\
&\leq ||f_j(X) - f_\infty(X)||_\infty \int_{B_r(Y)} |h(X)|dX \\
&\leq \varepsilon
\end{aligned}$$

$T_h(f_j)$ converges to $T_h(f_\infty)$. Hence, one has $T_h \in \mathcal{D}'(\mathbb{R}^n)$ and the order of the distribution is zero.

2. $\underline{\delta_Y \in \mathcal{D}'(\mathbb{R}^n)}$. Given that $\delta_Y := f(Y)$, for some $\lambda \in \mathbb{K}$ and $f_1, f_2 \in \mathcal{D}(\mathbb{R}^n)$, one has

$$\delta_Y(f_1 + \lambda f_2) = f_1(Y) + \lambda f_2(Y) = \delta_Y(f_1) + \lambda \delta_Y(f_2).$$

Thus, δ_Y is linear. Let us fix $\varepsilon > 0$ and $\alpha = (0, 0, \ldots, 0)$. By **Definition 1.1.6**, there exists $N \in \mathbb{N}$ such that for $j \geq N$.

$$||f_j - f_\infty||_\infty \leq \varepsilon$$

Then

$$\begin{aligned} |\delta_Y(f_j) - \delta_Y(f_\infty)| &= |f_j(Y) - f_\infty(Y)| \\ &\leq ||f_j(X) - f_\infty(X)||_\infty \\ &\leq \varepsilon \end{aligned}$$

Therefore, $\delta_Y(f_j)$ converges to $\delta_Y(f_\infty)$. Since δ_Y satisfies the linearity and convergence property, one has $\delta_Y \in \mathcal{D}'(\mathbb{R}^n)$. Since we fixed $\alpha = (0, 0, \ldots, 0)$, then zero is the order of the distribution.

3. $\underline{\delta_Y^\alpha \in \mathcal{D}'(\mathbb{R}^n)}$. Given that $\delta_Y^\alpha := (-1)^{|\alpha|}[\partial^\alpha f](Y)$, for some $\lambda \in \mathbb{K}$ and $f_1, f_2 \in \mathcal{D}(\mathbb{R}^n)$, one has

$$\delta_Y^\alpha(f_1 + \lambda f_2) = (-1)^{|\alpha|}[\partial^\alpha f_1](Y) + \lambda(-1)^{|\alpha|}[\partial^\alpha f_2](Y) = \delta_Y^\alpha(f_1) + \lambda \delta_Y^\alpha(f_2).$$

Thus, δ_Y^α is linear. Let us fix $\varepsilon > 0$ and by **Definition 1.1.6**, there exists $N \in \mathbb{N}$ such that for $j \geq N$.

$$||\partial^\alpha f_j - \partial^\alpha f_\infty||_\infty \leq \varepsilon$$

Then

$$\begin{aligned} |\delta_Y^\alpha(f_j) - \delta_Y^\alpha(f_\infty)| &= \left|(-1)^{|\alpha|}\partial^\alpha f_j(Y) - (-1)^{|\alpha|}\partial^\alpha f_\infty(Y)\right| \\ &= |\partial^\alpha f_j(Y) - \partial^\alpha f_\infty(Y)| \\ &\leq ||\partial^\alpha f_j(X) - \partial^\alpha f_\infty(X)||_\infty \\ &\leq \varepsilon \end{aligned}$$

Therefore, $\delta_Y^\alpha(f_j)$ converges to $\delta_Y^\alpha(f_\infty)$ and one has $\delta_Y^\alpha \in \mathcal{D}'(\mathbb{R}^n)$. The order of the distribution is given by $|\alpha|$ as we have $\alpha \in \mathbb{N}^n$.