Report 3

Mittwoch, 5. Juli 2023 09:33

Exercise 3.2.11.: i) Exhibit an example of strongly but not uniformly convergencent...
ii) Exhibit an example of weakly but not strongly convergent...
bounded operators.

Definition (Hilbertspace): A (complex) Hilbertspace $\mathcal{H}$ is a vector space on $\mathbb{C}$ with a strictly positive scalar product which is complete for the associated norm and which admits a countable orthonormal basis. (scalar product $\langle \cdot, \cdot \rangle$, norm $\| \cdot \|$)

Definition (Bounded linear operators): A map $B: \mathcal{H} \to \mathcal{H}$ is a bounded linear operator if $B: \mathcal{H} \to \mathcal{H}$ is a linear map and if $\exists\, c > 0$, s.t. $\|Bf\| \le c\|f\| \ \forall f \in \mathcal{H}$. The set of all bounded linear operators on $\mathcal{H}$ is denoted by $\mathscr{B}(\mathcal{H})$.

$$\|B\| := \inf\{c > 0 \mid \|Bf\| \le c\|f\| \ \forall f \in \mathcal{H}\} = \sup_{0 \ne f \in \mathcal{H}} \frac{\|Bf\|}{\|f\|}$$

Definition (Multiplication operator): If $\mathcal{H} = L^2(\mathbb{R}^n)$ (square integrable functions) and $m: \mathbb{R}^n \to \mathbb{C}$ a continous and bounded function, then $[Mf](x) := m(x) f(x) \ \forall f \in \mathcal{H} \wedge x \in \mathbb{R}^n$ defines a bounded linear operator on $\mathcal{H}$.

Definition (Convergence in $\mathscr{B}(\mathcal{H})$): Consider a sequence $(B_j)_{j \in \mathbb{N}} \subset \mathscr{B}(\mathcal{H})$, and let $B_\infty \in \mathscr{B}(\mathcal{H})$.

1. This sequence is uniformly convergent to B_∞ if $\lim_{j\to\infty} \|B_j - B_\infty\| = 0$
2. This sequence is strongly convergent to B_∞ if $\forall f \in \mathcal{H} \Rightarrow \lim_{j\to\infty} \|(B_j - B_\infty) f\| = 0$
3. This sequence is weakly convergent to B_∞ if $\forall f, g \in \mathcal{H} \Rightarrow \lim_{j\to\infty} \langle g, (B_j - B_\infty) f\rangle = 0$

This is denoted by $u\text{-}\lim_{j\to\infty} B_j = B_\infty$, $s\text{-}\lim_{j\to\infty} B_j = B_\infty$, and $w\text{-}\lim_{j\to\infty} B_j = B_\infty$ respectively.

Remark: The above conditions for convergence coincide for finite $\mathcal{H}$.

i) Let $\mathcal{H}$ be any seperable Hilbertspace with $\{e_j\}_{j\in\mathbb{N}}$ an orthonormal basis.

$\forall f \in \mathcal{H}$, $\sum_{j\in\mathbb{N}} |\langle e_j, f\rangle|^2 < \infty$ (decomposition on the basis)

$\Rightarrow \sum_{j=N}^{\infty} |\langle e_j, f\rangle|^2 \to 0$ as $N \to \infty$ (otherwise the previous sum could not converge)

$B_j = P_{\text{span}\{e_1, \dots, e_j\}}$, then

$$\|B_j f - \mathbb{1} f\|^2 = \Big\| \sum_{k=1}^{j} \langle e_k, f\rangle e_k - \sum_{k=1}^{\infty} \langle e_k, f\rangle e_k \Big\|^2$$

$$= \Big\| \sum_{k=j+1}^{\infty} \langle e_k, f\rangle e_k \Big\|^2 = \sum_{k=j+1}^{\infty} |\langle e_k, f\rangle|^2 \to 0 \text{ as } j \to \infty \text{ thus,}$$

$s\text{-}\lim_{j\to\infty} B_j = \mathbb{1}$. But for any j, $\|B_j - \mathbb{1}\| = 1$, since $\|(B_j - \mathbb{1}) e_{j+1}\| = \|e_{j+1}\| = 1 \Rightarrow B_j$ does not converge to $\mathbb{1}$ in norm.

Ex.

$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$, $B_3 = P_{\text{span}\{e_1, e_2, e_3\}}$

$$B_3 f = \sum_{j=1}^{n=3} \langle f, e_j\rangle e_j$$

$$= \langle f, e_1\rangle e_1 + \langle f, e_2\rangle e_2 + \langle f, e_3\rangle e_3$$

$$= \underbrace{(f_1 1 + f_2 0 + f_3 \cdot 0)}_{\in \mathbb{R}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\in \mathbb{R}^3} + f_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + f_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ f_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = f$$

ii) Let $\mathcal{H} = \ell^2(\mathbb{N})$, $f = \{f_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ with $\sum_{k\in\mathbb{N}} |f_k|^2 < \infty$

$B_j(\{f_k\}) = \begin{cases} f_{k+1} & \text{if } k > j \\ 0 & \text{if } k \le j \end{cases}$. This operator defines a right shift by filling up with 0's on the left side.

weak convergence:

$$|\langle g, B_j f\rangle|$$
$$= |\langle (g_1, g_2, \dots), (\overbrace{0, \dots, 0}^{j\text{-times}}, f_1, f_2, \dots)\rangle|$$
$$= |\langle (g_{j+1}, g_{j+2}, \dots), (f_1, f_2, \dots)\rangle|$$
$$= |\langle (g_{j+1}, g_{j+2}, \dots), f\rangle|$$
$$\Rightarrow \Big|\sum_{k=1}^{\infty} \langle g_{j+k}, f_k\rangle\Big| \overset{CS}{\le} \Big(\sum_{k=j+1}^{\infty} |g_k|^2\Big)^{\frac{1}{2}} \|f\| \to 0 \text{ as } j \to \infty \text{, since}$$
$$\sum_{k=1}^{\infty} |g_k|^2 < \infty$$

strong converence:

Let $f := e_1 \in \ell^2(\mathbb{Z})$

$$\lim_{j\to\infty} \|(B_j - B_\infty) f\|$$
$$\Leftrightarrow \lim_{j\to\infty} \|B_j f - B_\infty f\|$$
$$\Leftrightarrow \lim_{j\to\infty} \|e_j - 0\| = 1 \Rightarrow \exists f \text{ s.t. } \|(B_j - B_\infty) f\| \overset{\text{strongly}}{\not\to} 0$$

since B_j is norm preserving ($B_j f = (\underbrace{0, \dots, 0}_{j\text{-times}}, f_1, f_2, \dots) = (0_j, f)$
$\Rightarrow \|f\| = \|B_j f\|$

Ex. $B_1 e_1 = e_2$
$B_2 e_1 = e_3$
$\vdots$
$B_n e_1 = e_{n+1} = (\underbrace{0, \dots, 0}_{n\text{-times}}, 1, 0, \dots)$

Exercise 3.2.12.:

$$\lim_{j\to\infty} \|(B_j - B_\infty) f\| = 0$$
$$\lim_{j\to\infty} \|(A_j - A_\infty) f\| = 0 \quad \forall f \in \mathcal{H}$$
$$\|AB\| \le \|A\| \, \|B\| \ (*)$$

Claim: $(A_j B_j)_{j\in\mathbb{N}}$ is strongly convergent to the element $A_\infty B_\infty$ for $(A_j), (B_j) \subset \mathscr{B}(\mathcal{H})$ strongly convergent to A_∞, B_∞ respectively.

Note: $Bf \in \mathcal{H}$, $Af \in \mathcal{H}$, $ABf \in \mathcal{H}$ and $(A_j - A_\infty) B_j + A_\infty (B_j - B_\infty)$
$\Leftrightarrow A_j B_j - A_\infty B_j + A_\infty B_j - A_\infty B_\infty$

Proof: $\lim_{j\to\infty} \|(A_j B_j - A_\infty B_\infty) f\|$

$$\Leftrightarrow \lim_{j\to\infty} \|[(A_j - A_\infty) B_j + A_\infty (B_j - B_\infty)] f\|$$
$$\Leftrightarrow \lim_{j\to\infty} \|(A_j - A_\infty) B_j f + A_\infty (B_j - B_\infty) f\|$$
$$\Leftrightarrow \lim_{j\to\infty} \|(A_j - A_\infty)(B_j - B_\infty + B_\infty) f + A_\infty (B_j - B_\infty) f\|$$
$$\Leftrightarrow \lim_{j\to\infty} \|(A_j - A_\infty)[(B_j - B_\infty) f + B_\infty f] + A_\infty (B_j - B_\infty) f\|$$
$$\Leftrightarrow \lim_{j\to\infty} \|(A_j - A_\infty)(B_j - B_\infty) f + (A_j - A_\infty) B_\infty f + A_\infty (B_j - B_\infty) f\|$$
$$\le \lim_{j\to\infty} \|(A_j - A_\infty)(B_j - B_\infty) f\| + \|(A_j - A_\infty) B_\infty f\| + \|A_\infty (B_j - B_\infty) f\|$$
$$\overset{(*)}{\le} \lim_{j\to\infty} \underbrace{\|A_j - A_\infty\|}_{<\infty} \underbrace{\|(B_j - B_\infty) f\|}_{\to 0} + \underbrace{\|(A_j - A_\infty) \underbrace{B_\infty f}_{\text{element of } \mathcal{H}}\|}_{\to 0} + \underbrace{\|A_\infty (B_j - B_\infty) f\|}_{\to 0}$$

since $\|A_j - A_\infty\| < c$, c independent of j, after the uniform boundedness principle (*)

$\Rightarrow (A_j B_j)_{j\in\mathbb{N}}$ is strongly convergent to the element $A_\infty B_\infty$

(*) http://mathonline.wikidot.com/corollaries-to-the-uniform-boundedness-principle