Functional Analysis

Report

Chenxi Zeng

July 26, 2023

Problem 1

Exercise 3.3.13

For the operator $A, \forall f \in \mathscr{H}$ and any family $\{g_j, h_j\}_{j=1}^N \subset \mathscr{H}$, defined by

$$Af \equiv (\sum_{j=1}^{N} |h_j\rangle\langle g_j|)f := \sum_{j=1}^{N} \langle g_j, f\rangle h_j$$

give an upper estimate for $\|A\|$ and compute A^*.

Solution.

Clearly A is a bounded linear operator. Then

$$\|Af\| = \|\sum_{j=1}^{N} \langle g_j, f\rangle h_j\| \leq \sum_{j=1}^{N} |\langle g_j, f\rangle| \, \|h_j\| \leq \sum_{j=1}^{N} \|g_j\| \, \|h_j\| \, \|f\|$$

so we can get

$$\|A\| \leq \sum_{j=1}^{N} \|g_j\| \, \|h_j\|$$

and $\forall f, l \in \mathscr{H}$,

$$\langle l, Af\rangle = \sum_{j=1}^{N} \langle g_j, f\rangle\langle l, h_j\rangle = \langle \sum_{j=1}^{N} \langle h_j, l\rangle g_j, f\rangle = \langle A^* l, f\rangle$$

then

$$A^* = \sum_{j=1}^{N} |g_j\rangle\langle h_j|$$

Obviously A^* is also a finite rank operator.

□

Problem 2

Exercise 3.3.15

Check that a projection P is a compact operator if and only if $P\mathscr{H}$ is of finite dimension.

Solution.

1. Let $\dim(P\mathscr{H}) = N < \infty$. Then P is a finite rank operator. And obviously P is a compact operator(just let the $A_j = P, \forall j \in \mathbb{N}$).

2. Using the proof by contradiction, we need prove that: If $\dim(\mathscr{M} := P\mathscr{H}) = \infty$, then $P \notin \mathscr{K}(\mathscr{H})$.

Let $\dim(\mathscr{M}) = \infty$. Then for any finite rank operator A, generally defined in Problem 1, that $Ran(A) \subset Vect(h_1, ..., h_N)$, there must exist an vector $a \neq 0, a \in \mathscr{M}$, such that $\langle a, h_j\rangle = 0, 1 \leq j \leq N$. Then $\langle a, Aa\rangle = 0$. So

$$||P - A|| = \sup_{0 \neq f \in (\mathscr{H})} \frac{||(P - A)f||}{||f||} \geq \frac{\sqrt{\langle a - Aa, a - Aa \rangle}}{||a||} = \frac{\sqrt{||a||^2 + ||Aa||^2}}{||a||} \geq 1$$

which means that for any finite rank operator A, $||P - A|| \geq 1$. Then there doesn't exist a sequence of finite rank operator $\{A_j\}_{j \in \mathbb{N}}$ such that $lim_{j \to \infty} ||P - A_j|| = 0$. So $P \notin \mathscr{K}(\mathscr{H})$.

□

Problem 3

Proof the following properties for $\mathscr{K}(\mathscr{H})$:

1. $B \in \mathscr{K}(\mathscr{H}) \iff B^* \in \mathscr{K}(\mathscr{H})$.
2. $\mathscr{K}(\mathscr{H})$ is a *-algebra, complete for the norm$|| \cdot ||$.
3. If $B \in \mathscr{K}(\mathscr{H})$ and $A \in \mathscr{B}(\mathscr{H})$, then AB and BA belong to $\mathscr{K}(\mathscr{H})$.

Solution.

1. If $B \in \mathscr{K}(\mathscr{H})$, then there exist a family $\{B_j\}_{j \in \mathbb{N}}$, such that $lim_{j \to \infty} ||B - B_j|| = 0$. Then $lim_{j \to \infty} ||B^* - B_j^*|| = 0$. We can see in Problem 1 that B_j^* is also a finite operator. So $B^* \in \mathscr{K}(\mathscr{H})$. Use the relation $(B^*)^* = B$ can proof the other half.

2. $\mathscr{K}(\mathscr{H})$ is clearly a vector space. Algebra, a vector space equipped with an operation of multiplication. The product of two compact operators is just the composition of two mappings: $ABf = A(Bf)$. And from the $B \in \mathscr{K}(\mathscr{H}) \iff B^* \in \mathscr{K}(\mathscr{H})$ we know that $\mathscr{K}(\mathscr{H})$ is involutive, called a *-algebra.

If there is a sequence A_n in $\mathscr{K}(\mathscr{H})$, $j \in \mathbb{N}$, such that $lim_{n \to \infty} ||A - A_n|| = 0$. Then $\forall \epsilon > 0$, $\exists n$, such that $||A - A_n|| < \frac{\epsilon}{2}$. And there exists a family$\{A_{nj}\}_{j \in \mathbb{N}}$ of finite rank operators such that $lim_{j \to \infty} ||A_n - A_{nj}|| = 0$, then for the ϵ above, $\exists M \in \mathbb{N}$, such that $\forall j > M$, $||A_n - A_{nj}|| < \frac{\epsilon}{2}$. Then we get

$$\forall \epsilon > 0, \exists n, M \in \mathbb{N},\ such\, that\, \forall j > M,\ ||A - A_{nj}|| \leq ||A - A_n|| + ||A - A_{nj}|| = \epsilon$$

which means $lim_{j \to \infty} ||A - A_{nj}|| = 0$, A_{nj} is finite operator. So $A \in \mathscr{K}(\mathscr{H})$, $\mathscr{K}(\mathscr{H})$ is complete for the norm $|| \cdot ||$.

3. For $B \in \mathscr{K}(\mathscr{H})$, there exists a family $\{B_{nj}\}_{j \in \mathbb{N}}$ such that $lim_{j \to \infty} ||B - B_j|| = 0$. Obviously AB_j, B_jA are finite operators, and

$$||AB - AB_j|| \leq |||A|| \, ||B - B_j||$$

$$||BA - B_jA|| \leq ||B - B_j|| \, ||A||$$

then $lim_{j \to \infty} ||AB - AB_j|| = lim_{j \to \infty} ||BA - B_jA|| = 0$, AB and $BA \in \mathscr{K}(\mathscr{H})$.

□