1 Exercise 1.3.8

Exercise 1.3.8. *Consider $h : \mathbb{R}^n \to \mathbb{K}$ satisfying $\int_{\mathbb{R}^n} |h(X)|\, \mathrm{d}X < \infty$, and assume that $\int_{\mathbb{R}^n} h(X)\, \mathrm{d}X = 1$. For $j \in \mathbb{N}$, set $h_j(X) := j^n h(jX)$. Then, prove that $T_{h_j} \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $j \to \infty$. Equivalently, for $\varepsilon > 0$ one often sets $h_\varepsilon(X) := \frac{1}{\varepsilon^n} h\left(\frac{X}{\varepsilon}\right)$. Show that $T_{h_\varepsilon} \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.*

Theorem 1.1 (Triangle Inequality)**.** *The sum of any two sides of a triangle is greater than or equal to the third side.*

$$|A + B| \leq |A| + |B|$$

We begin with the definition of a limit.

Definition 1.1 (The limit of a sequence)**.** *The limit $\lim_{j\to\infty} a_j = \alpha$ exists if $\forall \epsilon > 0, \exists J \in \mathbb{N}$ such that $|a_j - \alpha| < \epsilon, \forall j > J$.*

Proof. For $T_{h_j}, \delta_0 \in \mathcal{D}'(\mathbb{R}^n)$ and $f \in \mathcal{D}(\mathbb{R}^n)$.

The equality $\lim_{j\to\infty} T_{h_j} = \delta_0$ holds if $\forall \epsilon > 0, \exists J \in \mathbb{N}$ such that $|T_{h_j}(f) - \delta_0(f)| < \epsilon, \forall j > J$.

$$\begin{aligned} &|T_{h_j}(f) - \delta_0(f)| \\ &= |\int_{\mathbb{R}^n} j^n h_j(\frac{X}{j}) f(X)\, dX - \int_{\mathbb{R}^n} \delta_0(X) f(X)\, dX| \end{aligned}$$

Note: $\int_{\mathbb{R}^n} \delta_0(X) f(X)\, dX = f(0) = f(0) \int_{\mathbb{R}^n} h(X)\, dX$ since $\int_{\mathbb{R}^n} h(X)\, dX = 1$

$$= |\int_{\mathbb{R}^n} j^n h_j(\frac{X}{j}) f(X)\, dX - \int_{\mathbb{R}^n} h(X) f(0)\, dX|$$

Change of Variables: Let $X = jY$

$$dX = j^n dY$$

$$= |\int_{\mathbb{R}^n} h(Y) f(\frac{Y}{j})\, dY - \int_{\mathbb{R}^n} h(X) f(0)\, dX|$$

Match the Variables

$$\begin{aligned} &= |\int_{\mathbb{R}^n} h(\theta) f(\frac{\theta}{j})\, d\theta - \int_{\mathbb{R}^n} h(\theta) f(0)\, d\theta| \\ &= |\int_{\mathbb{R}^n} h(\theta)[f(\frac{\theta}{j}) - f(0)]\, d\theta| \end{aligned}$$

Consider two subsets defined by $\mathcal{B}_r(0) = \{X \in \mathbb{R}^n |\ ||X|| < r\}$

$$= |\int_{\mathcal{B}_r(0)} h(\theta)[f(\frac{\theta}{j}) - f(0)]\, d\theta + \int_{\mathbb{R}^n \setminus \mathcal{B}_r(0)} h(\theta)[f(\frac{\theta}{j}) - f(0)]\, d\theta|$$

$f(\frac{\theta}{j})$ becomes closer to $f(0)$ for $\theta \in \mathcal{B}_r(0)$ as j $\to \infty$

thereby we get, $f(\frac{\theta}{j}) - f(0) \to 0$ as j $\to \infty$, for $\theta \in \mathcal{B}_r(0)$

By the Triangle Inequality Theorem 1.1, we split the sum

$$\leq |\int_{\mathcal{B}_r(0)} h(\theta)[f(\frac{\theta}{j}) - f(0)]\, d\theta| + |\int_{\mathbb{R}^n \setminus \mathcal{B}_r(0)} h(\theta)[f(\frac{\theta}{j}) - f(0)]\, d\theta|$$

$$\leq \int_{\mathcal{B}_r(0)} |h(\theta)||f(\frac{\theta}{j}) - f(0)|\, d\theta + \int_{\mathbb{R}^n \setminus \mathcal{B}_r(0)} |h(\theta)||f(\frac{\theta}{j}) - f(0)|\, d\theta$$

Introduce $||f||_\infty := \sup_{\theta' \in \mathbb{R}^n} |f(\theta')|$

$$\leq \sup_{\theta' \in \mathcal{B}_r(0)} |f(\frac{\theta'}{j}) - f(0)| \int_{\mathcal{B}_r(0)} |h(\theta)|\, d\theta + 2||f||_\infty \int_{\mathbb{R}^n \setminus \mathcal{B}_r(0)} |h(\theta)|\, d\theta$$

Now we choose:

1. r such that $\int_{\mathbb{R}^n \setminus \mathcal{B}_r(0)} |h(\theta)|\, d\theta < \frac{\epsilon}{4||f||_\infty}$
2. J such that $\sup_{\theta' \in \mathcal{B}_r(0)} |f(\frac{\theta'}{j}) - f(0)| < \frac{\epsilon}{2\int_{\mathbb{R}^n} |h(\theta)|\, d\theta}$, $\forall j > J$

$$\leq \frac{\epsilon}{2} \frac{\int_{\mathcal{B}_r(0)} |h(\theta)|\, d\theta}{\int_{\mathbb{R}^n} |h(\theta)|\, d\theta} + \frac{\epsilon}{2}$$

$$\leq \epsilon$$

□

2 Exercise 1.3.9

Exercise 1.3.9. *For $j \in \mathbb{N}$, consider $h_j(x) := \frac{\sin(jx)}{x}$ for any $x \in \mathbb{R} \setminus \{0\}$ and $h_j(0) := j$. Show that $T_{h_j} \to \pi\delta_0$ in $\mathcal{D}'(\mathbb{R})$ as $j \to \infty$. Why is this situation not covered by the previous exercise ?*

The condition for the situation in the previous exercise was for a distribution $T_h \in L^1(\mathbb{R}^n)$, i.e. $\int_{\mathbb{R}^n} |h(X)|\, dX < \infty$. However, in Exercise 1.3.9, we have $h(X) := \frac{sin(X)}{X}$ where $\int_{\mathbb{R}^n} |h(X)|\, dX = \infty$. $h \in L^1_{loc}(\mathbb{R})$ as opposed to $L^1(\mathbb{R})$.

Before we begin, I'd like to present a few theorems and lemmas.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous and $g : I \to [a,b]$ be continuously differentiable with $Im(g) \subset [a,b]$, where $I \subset \mathbb{R}$ is some open interval. Then, the function*

$$F(x) := \int_a^{g(x)} f(t)\, dt$$

is continuously differentiable on I and

$$F'(x) = g'(x)f(g(x))$$

Theorem 2.2 (Dominated Convergent Theorem)**.** *If $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is a sequence of measurable functions which converge point-wise almost everywhere to f, and if there exists an integrable function g such that $|f_n(x)| \leq g(x)$ $\forall n$ and $\forall x$, then f is integrable and*

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n$$

Proof. Let us begin by setting

$$F(x) := \lim_{n\to-\infty} \int_n^x \frac{sin(t)}{t}\, dt$$

This integral is well-defined since it is an integral of a continuous and even function and it exists as an improper Riemann integral where $\int_0^\infty \frac{sin(t)}{t}\, dt = \frac{\pi}{2}$.

Observation 2.1.

$$\lim_{x\to-\infty} F(x) = 0 \text{ and } \lim_{x\to\infty} F(x) = \pi$$

Via Lemma 2.1, since $h(x) = \frac{sin(x)}{x}$ is continuous, F is differentiable with $F'(x) = \frac{sin(x)}{x}$.

Observation 2.2.

$$\frac{d}{dx}F(jx) = F'(jx)j = \frac{sin(jx)j}{jx} = \frac{sin(jx)}{x}$$

$$\begin{aligned}\text{Then } T_{h_j}(f) &= \int_{\mathbb{R}} \frac{sin(jx)}{x} f(x)\, dx \\ &= \int_{\mathbb{R}} F'(jx) f(x)\, dx\end{aligned}$$

By integration by parts

$$\begin{aligned}&= F(jx)f(x)|_{-\infty}^{\infty} - \int_{\mathbb{R}} F(jx) f'(x)\, dx \\ &= -\int_{\mathbb{R}} F(jx) f'(x)\, dx\end{aligned}$$

Observation 2.3. *Since F is continuous:*

$$\lim_{x\to-\infty} F(x) = 0 \text{ and } \lim_{x\to\infty} F(x) = \pi$$

It follows that F is bounded, namely, $||F||_\infty < \infty$
Furthermore,

$$|\int_{\mathbb{R}} F(jx) f'(x)\, dx| \le ||F||_\infty \int_{\mathbb{R}} |f'(x)|\, dx < C$$

where C is independent of j.

We can apply Theorem 2.2, and exchange the limit and the integral.

$$\lim_{j\to\infty} -\int_{\mathbb{R}} F(jx) f'(x)\, dx = -\int_{\mathbb{R}} \lim_{j\to\infty} F(jx) f'(x)\, dx$$

Observation 2.4. *For a fixed $x \neq 0$: $\lim_{j\to\infty} F(jx) = \begin{cases} \pi & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$*

$$\begin{aligned}-\int_{\mathbb{R}} \lim_{j\to\infty} F(jx) f'(x)\, dx &= -\pi \int_0^\infty f'(x)\, dx \\ &= -\pi f(x)|_0^\infty \\ &= \pi f(0)\end{aligned}$$

Thus we obtain

$$\lim_{j\to\infty} T_{h_j} = \pi\delta_0$$

$\square$