Reminder XIV

Fourier

- Differential operators : $((-i\partial)^\alpha, \mathcal{F}^* D(X^\alpha))$ is self-adjoint. Based on the equality $\mathcal{F}(-i\partial_j)f = X_j \mathcal{F} f$.

Laplace operator

$(-\Delta, \mathcal{F}^* D(X^2)) = (-\Delta, \mathcal{F}^* D(\langle X \rangle^2)) = (-\Delta, \mathcal{H}^2(\mathbb{R}^n))$

is self-adjoint

2^{nd} Sobolev Space

$X = (X_1, \dots X_n)$ = position operators, $D = (D_1, \dots, D_n)$ = momentum operators, with $D_j = -i\partial_j$

- Convolution operator : $\varphi(D) := \mathcal{F}^* \varphi(X) \mathcal{F}$ with $D(\varphi(D)) = \mathcal{F}^* \varphi(X)$.

- Spectral family : $E : \mathbb{R} \to \mathcal{P}(\mathcal{H})$ satisfying

$E_\mu E_\lambda = E_{\min(\mu,\lambda)}$, $E_{\lambda+0} = E_\lambda$, $E_{-\infty} = 0$, $E_\infty = 1$.

One-to one correspondence with spectral measure.

$\rightsquigarrow$ For $\varphi : \mathbb{R} \to \mathbb{C}$, measurable, Riemann type integral $\int_{\mathbb{R}} \varphi(\lambda)\, E(d\lambda)$, with $D_\varphi =$

$= \{ f \in \mathcal{H} \mid \int_{\mathbb{R}} |\varphi(\lambda)|^2 \langle E(d\lambda) f, f \rangle < \infty \}$

Thm (Spectral thm)

For any self-adjoint operator $(A, D(A))$, $\exists!$ spectral family (or spectral measure) such that

$A = \int_{\mathbb{R}} \lambda \, E(d\lambda)$ and $D(A) = \mathcal{D}_{id}$

↑ identity function $id(\lambda) = \lambda$

↗ this generalizes the diagonalization of Hermitian matrices.

Remark: Also a one-to-one correspondence with strongly continuous unitary groups.

↳ linked with dynamics.

Good luck for your future studies