DECOMPOSABILITY OF SCATTERING OPERATOR

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1. Direct integral

Let us recall the usual direct sum of Hilbert spaces. Let Y be a set and $\mathcal{H}_y$ be separable Hilbert spaces with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_y}$ for each $y \in Y$. The *algebraic direct sum* of $\{\mathcal{H}_y\}_{y \in Y}$, denoted by $\bigoplus_{y \in Y} \mathcal{H}_y$, is defined as the following vector space:

$$\bigoplus_{y \in Y} \mathcal{H}_y = \{(h_y)_{y \in Y} \in \Pi_{y \in Y} \mathcal{H}_y \mid \#(\mathrm{supp}(h_y)) < \infty\},$$

where $\mathrm{supp}(h_y) = \{y \in Y \mid h_y \neq 0\}$ and $\Pi_{y \in Y} \mathcal{H}_y = \{(h_y)_{y \in Y} \mid h_y \in \mathcal{H}_y\}$. This space is also the inner product space with $\langle \cdot, \cdot \rangle = \sum_{y \in Y} \langle \cdot, \cdot \rangle_{\mathcal{H}_y}$. Here, we define the *Hilbert space direct sum* of $\{\mathcal{H}_y\}_{y \in Y}$, denoted by $\overline{\bigoplus}_{y \in Y} \mathcal{H}_y$ as the completion of $\bigoplus_{y \in Y} \mathcal{H}_y$ with respect to the inner product $\langle \cdot, \cdot \rangle$, that is,

$$\overline{\bigoplus_{y \in Y}} \mathcal{H}_y = \left\{ (h_y)_{y \in Y} \in \Pi_{y \in Y} \;\middle|\; \sum_{y \in Y} \|h_y\|_{\mathcal{H}_y}^2 < \infty \right\}.$$

Their inner product is also denoted by $\langle , \rangle$. If $\mathcal{H}_y = \mathcal{H}$ for all $y \in Y$ for some a separable Hilbert space $\mathcal{H}$, then we can also see the $\overline{\bigoplus}_{y \in Y} \mathcal{H}_y$ as follows: Set $\mathfrak{A} = 2^Y$, that is, the trivial (most strong) σ-algebra of Y and let μ be the discrete measure (counting measure) on measurable space $(Y, \mathfrak{A})$. Then, we have

$$\overline{\bigoplus_{y \in Y}} \mathcal{H}_y = \left\{ h := (h_y)_{y \in Y} \in \Pi_{y \in Y} \;\middle|\; \|h\|^2 := \int_Y \|h_y\|_{\mathcal{H}}^2 \, d\mu < \infty \right\} = l^2(Y; \mathcal{H}),$$

the right hand side is the $\mathcal{H}$-valued l^2-space on Y (the space of L^2-functions on the discrete measure space $(Y, \mathfrak{A}, \mu)$). By the way, how is the case when $\mathcal{H}_y \neq \mathcal{H}_{y'}$ for some $y, y' \in Y$? If $\mathrm{ran}(\mathcal{H}_{(\cdot)}) = \{\mathcal{H}[\lambda]\}_{\lambda \in \Lambda}$ for some set Λ and we define $Y_\lambda := \{y \in Y \mid \mathcal{H}_y = \mathcal{H}[\lambda]\}$, then it is easily seen that

$$\overline{\bigoplus_{y \in Y}} \mathcal{H}_y = \overline{\bigoplus_{\lambda \in \Lambda}} l^2(Y_\lambda; \mathcal{H}[\lambda]).$$

Also, how is the case when $(Y, \mathfrak{A}, \mu)$ is not discrete ? It is easy to guess that the resulting spaces become (the Hilbert spaces direct sum of) the spaces of L^2-functions which take value in different Hilbert spaces $\mathcal{H}_y$ for each $y \in Y$. But, for that, we need some appropriate measurability as seen bellow.

Let $(Y, \mathfrak{A}, \mu)$ be a (separable) σ-finite measure space. Let also $\mathcal{H}$ be a function which is defined on μ-a.e. Y and each value $\mathcal{H}(y)$ is a separable Hilbert space for μ-a.e. $y \in Y$.

Definition 1.1. Let Ω_0 be a finite or countable set of $\bigsqcup_{y \in Y} \mathcal{H}(y)$-valued functions whose value at $y \in Y$ in $\mathcal{H}(y)$, defined on μ-a.e. Y. The set Ω_0 is the *base of measurability* if Ω_0 satisfies

(1) $\overline{\mathrm{span}}\{g(y) \mid g \in \Omega_0\} = \mathcal{H}(y)$ for μ-a.e. $y \in Y$, where $\overline{\mathrm{span}}$ is the closure of linear span.
(2) $\langle g_1(\cdot), g_2(\cdot) \rangle_{\mathcal{H}(\cdot)}$ is a μ-measurable function defined on μ-a.e. Y for any $g_1, g_2 \in \Omega_0$.

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A $\bigsqcup_{y\in Y}\mathcal{H}(y)$-valued function h such that $h(y)\in\mathcal{H}(y)$ is called *measurable* with respect to Ω_0 if for any $g\in\Omega_0$, $\langle h(\cdot),g(\cdot)\rangle_{\mathcal{H}(\cdot)}$ is μ-measurable. We denote by $\widehat{\Omega_0}$ denotes the set of all measurable functions with respect to Ω_0. The family of Hilbert spaces $\{\mathcal{H}(y)\}_{y\in Y}$ endowed with a measurable structure $\widehat{\Omega_0}$ is called a *measurable Hilbert family* on the measure space $(Y,\mathfrak{A},\mu)$, which is denoted by $(\mathcal{H}(\cdot),\widehat{\Omega_0})$.

Remark 1.2. It is clear that $\widehat{\Omega_0}\supset\Omega_0$. Also, $\widehat{\Omega_0}$ is invariant under multiplication by $\mathbb{C}$-valued μ-measurable function.

Now, in a position to define the direct integral of $\mathcal{H}$ with respect to $(Y,\mathfrak{A},\mu)$.

Definition 1.3. The Ω_0 be a base of measurability. Then, we define the *direct integral* of $\mathcal{H}$ with respect to Ω_0 as follows:

$$H:=\int_Y^{\oplus\widehat{\Omega_0}}\mathcal{H}(y)\,d\mu(y):=\left\{h\in\widehat{\Omega_0}\,\middle|\,\|h\|^2:=\int_Y\|h(y)\|^2_{\mathcal{H}(y)}\,d\mu(y)<\infty\right\}$$

with inner product

$$\langle g,h\rangle:=\int_Y\langle g(y),h(y)\rangle_{\mathcal{H}(y)}\,d\mu$$

for any $g,h\in H$. Now, we can know only the direct integral is pre-Hilbert space. In fact, we also can see the direct integral is complete (see Example 1.13).

In the sequel, we study some basic notion related to direct integral. In general, the choice of the base of measurability is not unique, so that we may take better one in some sense. The following lemma shows that we can always take an orthonormal base as base of measurability: Let $m:=\mu\text{-}\sup\{\dim(\mathcal{H}(y))\,|\,y\in Y\}$, $1\le m\le\infty$ and $\Omega_1=\{e_j\}_{j\in[1,m\rangle}$, where $\{e_j(y)\}_{j\in[1,\dim(\mathcal{H}(y))\rangle}$ is an orthonormal base in $\mathcal{H}(y)$ and $e_j(y)=0$ for $j>\dim(\mathcal{H}(y))$ (here, if $k<\infty$, then $[1,k\rangle=\{1,2,\cdots,k\}$; if $k=\infty$, then $[1,k\rangle=\mathbb{Z}_{>0}$). Clearly, Ω_1 satisfies the two properties of Definition 1.1, which is called the *orthogonal base of measurability.*

Lemma 1.4. Let Ω_0 be a base of measurability. Then, there exists an orthogonal base of measurability $\Omega_1\subset\widehat{\Omega_0}$ such that $\widehat{\Omega_1}=\widehat{\Omega_0}$.

Proof. Use Gram-Schmidt orthogonalization. □

By the proof of Lemma 1.4, we have the following fact:

Corollary 1.5. For any $g,h\in\widehat{\Omega_0}$, the function $\langle g(\cdot),h(\cdot)\rangle_{\mathcal{H}(\cdot)}$ is μ-measurable.

Let $(\mathcal{H}(\cdot),\widehat{\Omega_0})$ and $(\mathcal{H}'(\cdot),\widehat{\Omega_0'})$ be two measurable Hilbert families on $(Y,\mathfrak{A},\mu)$. Then, we can define a measurable structure in the set of operator-valued function F whose values $F(y)$ belongs to $B(\mathcal{H}(y),\mathcal{H}'(y))$ μ-a.e.

Definition 1.6. The function F as above is *measurable* if $\langle F(\cdot)h(\cdot),h'(\cdot)\rangle_{\mathcal{H}'(\cdot)}$ are measurable for any $h\in\widehat{\Omega_0}$ and any $h'\in\widehat{\Omega_0'}$. This is equivalent to the condition: $\langle F(\cdot)h(\cdot),h'(\cdot)\rangle_{\mathcal{H}'(\cdot)}$ are measurable for any $h\in\Omega_0$ and $h'\in\Omega_0'$. Here, it is clear that F is measurable if and only if so is F^*.

Two Hilbert spaces which have the same dimension are unitarily isomorphic, so that two Hilbert space direct sums whose corresponding component have same dimension are also unitarily isomorphic. Similarly, two measurable Hilbert families $(\mathcal{H},\widehat{\Omega_0})$ and $(\mathcal{H}',\widehat{\Omega_0'})$ such that $\dim(\mathcal{H}(y))=\dim(\mathcal{H}'(y))$ for μ-a.e. $y\in Y$ are unitarily isomorphic. We leave it as the following lemma:

Lemma 1.7. Let us consider two measurable Hilbert families $(\mathcal{H},\widehat{\Omega_0})$ and $(\mathcal{H}',\widehat{\Omega_0'})$ which satisfy that $\dim(\mathcal{H}(y))=\dim(\mathcal{H}'(y))$ for μ-a.e. $y\in Y$. Then, there exists a measurable *unitary* operator-valued function w whose value $w(y)$ are from $\mathcal{H}(y)$ onto $\mathcal{H}'(y)$ μ-a.e. $y\in Y$ such that $\widehat{\Omega_0'}=\{wh\,|\,h\in\widehat{\Omega_0}\}$.

Proof. By Lemma 1.4, we can assume that Ω_0 and Ω_0' are orthogonal bases of measurability. Hence, if we define $w : (\mathcal{H}, \widehat{\Omega_0}) \to (\mathcal{H}', \widehat{\Omega_0'})$ by

$$(we_j)(y) := e_j'(y)$$

for μ-a.e. $y \in Y$, where $\Omega_0 = \{e_j\}_{j\in[1,m\rangle}$ and $\Omega_0' = \{e_j'\}_{j\in[1,m\rangle}$, $m = \mu\text{-}\sup\{\dim(\mathcal{H}(y))\}$. Then, w is the desired map. □

Hence, we have the same property for direct integrals as follows:

Corollary 1.8. Let us consider two measurable Hilbert families $(\mathcal{H}, \widehat{\Omega_0})$ and $(\mathcal{H}', \widehat{\Omega_0'})$ which satisfy that $\dim(\mathcal{H}(y)) = \dim(\mathcal{H}'(y))$ for μ-a.e. $y \in Y$. Let w be the unitary operator-valued function of Lemma 1.7. Then, $W : \int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y) \to \int_Y^{\oplus \widehat{\Omega_0'}} \mathcal{H}'(y)\, d\mu(y)$ defined by $(Wh)(y) := w(y)h(y)$ is a unitary isomorphism.

The next lemma provides a dense subset of $\int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y)$:

Lemma 1.9. Let $(\mathcal{H}, \widehat{\Omega_0})$ be a measurable Hilbert family on $(Y, \mathfrak{A}, \mu)$ and let $\{e_j\}_{j\in[1,m\rangle}$ be an orthogonal base of measurability for $\widehat{\Omega_0}$, where $m = \mu\text{-}\sup\{\dim(\mathcal{H}(y))\}$ (that is, assume $\Omega_0 = \{e_j\}_{j\in[1,m\rangle}$). Let f_0 be an element of $L^2(Y, \mu; \mathbb{C})$ such that $f_0 \neq 0$ μ-a.e. Y. Then, the set of functions $h_{j,\delta}(y) = f_0(y)\chi_\delta(y)e_j(y)$, where $\delta \in \mathfrak{A}$, $j \in [1, m\rangle$ and χ_δ is the characteristic function on δ, is dense in $\int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y)$.

Proof. Let $g \in \int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y)$ such that $g \perp h_{j,\delta}$ for any $j \in [1, m\rangle$ and $\delta \in \mathfrak{A}$. Here, we have

$$\int_\delta \overline{f_0(y)} \langle g(y), e_j(y)\rangle_{\mathcal{H}(y)}\, d\mu(y) = \int_Y \langle g(y), h_{j,\delta}(y)\rangle_{\mathcal{H}(y)}\, d\mu(y) = \langle g, h_{j,\delta}\rangle = 0.$$

Since $\delta \in \mathfrak{A}$ is arbitrary, we have $\langle g(y), e_j(y)\rangle_{\mathcal{H}(y)} = 0$ for any $j \in [1, m\rangle$ and μ-a.e. $y \in Y$. This implies that

$$\langle g, e_j\rangle := \int_Y \langle g(y), e_j(y)\rangle_{\mathcal{H}(y)}\, d\mu(y) = 0.$$

for any $j \in [1, m\rangle$, and thus $g = 0$. □

The basic object usually considered in the context of direct integral is multiplication operator by a $\mathbb{C}$-valued function.

Definition 1.10. Let φ be a measurable $\mathbb{C}$-valued function on defined μ-a.e. Y. Then, the operator $\mathcal{Q}_\varphi$ in the direct integral is defined as the following operator with domain

$$\mathrm{dom}(\mathcal{Q}_\varphi) = \left\{ g \in \int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y) \,\middle|\, \int_Y |\varphi(y)|^2 \|g(y)\|^2_{\mathcal{H}(y)}\, d\mu(y) < \infty \right\}$$

such that

$$(\mathcal{Q}_\varphi g)(y) := \varphi(y)g(y)$$

for any $g \in \mathrm{dom}(\mathcal{Q}_\varphi)$.

The following corollary is clear:

Corollary 1.11. Let us consider two measurable Hilbert families $(\mathcal{H}, \widehat{\Omega_0})$ and $(\mathcal{H}', \widehat{\Omega_0'})$ which satisfy that $\dim(\mathcal{H}(y)) = \dim(\mathcal{H}'(y))$ for μ-a.e. $y \in Y$. Let also $\mathcal{Q}_\varphi$ and $\mathcal{Q}_\varphi'$ be the multiplication operator by some $\mathbb{C}$-valued function on $\int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y)$ and on $\int_Y^{\oplus \widehat{\Omega_0'}} \mathcal{H}'(y)\, d\mu(y)$, respectively. Then, we have $\mathcal{Q}_\varphi' W = W \mathcal{Q}_\varphi$, where W is the unitary operator of Corollary 1.8.

According to Corollary 1.8 and 1.11, the concrete choice of measurable Hilbert families is not so important. Hence, we can omit $\widehat{\Omega_0}$ in the symbol of the direct integral of $(\mathcal{H}(\cdot), \widehat{\Omega_0})$, that is,

$$\int_Y^{\oplus} \mathcal{H}(y)\, d\mu(y).$$

To end this section, we see some examples of direct integral.

Example 1.12. Let $(Y, \mathfrak{A}, \mu) = (\{x\}, \{\emptyset, \{x\}, \mu\})$ such that $\mu(\{x\}) = 1$ and $\mathcal{H}(x) = H$, where H is a separable Hilbert space. Let Ω_0 be an orthonormal base of H. Then it is clear that $(\mathcal{H}, \widehat{\Omega_0})$ is an orthogonal base of measurability and

$$\int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y) = \left\{h : \{x\} \to H \,\middle|\, \|h\|^2 = \|h(x)\|_H < \infty\right\} \simeq H.$$

Similarly, if $(Y, \mathfrak{A}, \mu) = (\{x_1, \cdots, x_n\}, 2^{\{x_1, \cdots, x_n\}}, \mu)$ such that $\mu(\{x_i\}) = 1$ and $\mathcal{H}(x_i) = H_i$, then we have

$$\int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y) = H_1 \oplus H_2 \oplus \cdots \oplus H_n.$$

Example 1.13. Let $(Y, \mathfrak{A}, \mu)$ is a σ-finite measure space and $\mathcal{H}(y) = H$ where H is a separable Hilbert space. Let Ω_0 be an orthonormal base of H. Then, it is easy to see that

$$\int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y) \simeq L^2(Y, \mu; H).$$

More generally, let $\mathcal{H}$ be a function whose value $\mathcal{H}(y)$ are separable Hilbert spaces and $Y = \bigsqcup_{\lambda \in \Lambda} Y_\lambda$ such that $Y_\lambda = \{y \in Y \mid \mathcal{H}(y) = H_\lambda\}$. If we set $\Omega_0 = \{e_j\}$ such that

$$e_j(y) = \begin{cases} e_j^\lambda, & \text{if } j \in [1, \dim(H_\lambda)\rangle,\ y \in Y_\lambda \\ 0, & \text{otherwise,} \end{cases}$$

where $\{e_j^\lambda\}_{j \in [1, \dim(H_\lambda)\rangle}$ is an orthonormal base of H_λ, then we have

$$\int_Y^{\oplus \widehat{\Omega_0}} \mathcal{H}(y)\, d\mu(y) \simeq \overline{\bigoplus_{\lambda \in \Lambda}} L^2(Y_\lambda, \mu|_{Y_\lambda}; H_\lambda).$$

By this example, we can see the direct integral is complete using Corollary 1.8.

2. Decomposable operator

Let $(Y, \mathfrak{A}, \mu)$ be a (separable) σ-finite measure space and H be an direct integral $\int_Y^\oplus \mathcal{H}(y)\, d\mu(y)$.

Definition 2.1. For $\delta \in \mathfrak{A}$, we define $X(\delta)$ as the operator from H to H by

$$(X(\delta)g)(y) := \chi_\delta(y) g(y)$$

for any $g \in H$.

It is easy to see that $X(\delta)$, $\delta \in \mathfrak{A}$, is the orthogonal projection onto the subspace of these vector-functions $g \in H$ which equal zero μ-a.e. on $Y \setminus \delta$ and $\{X(\delta)\}_{\delta \in \mathfrak{A}}$ defines a spectral measure on H. Then, for a measurable $\mathbb{C}$-valued function φ defined μ-a.e. on Y, we define an operator $\mathcal{J}_\varphi$ by spectral integral:

$$\mathcal{J}_\varphi := \int_Y \varphi(y)\, dX(y)$$

with the domain

$$\operatorname{dom}(\mathcal{J}_\varphi) := \left\{ g \in H \,\middle|\, \int_Y |\varphi(y)|^2\, d\langle X(y)g, g\rangle < \infty \right\}.$$

Here, we can easily see the following fact:

Corollary 2.2. The operator $\mathcal{J}_\varphi$ coincides with the operator $\mathcal{Q}_\varphi$.

Proof. See [BS, Chapter 7, Theorem 2.1]. □

Let $(Y, \mathfrak{A}, \nu)$ be another σ-finite measure space. We say that $\mu << \nu$ if for $\delta \in \mathfrak{A}$, $\nu(\delta) = 0$ implies $\mu(\delta) = 0$. We also say that $\mu \sim \nu$ if $\mu << \nu$ and $\nu << \mu$ hold. Note that for $\mathbb{C}$-valued function φ on Y, that φ is measurable with respect to μ is equivalent to that φ is measurable with respect to ν if $\mu \sim \nu$. Let H' be another direct integral $\int^{\oplus} \mathcal{H}'(y)\, d\nu(y)$, where $\mathcal{H}'(y)$ be separable Hilbert spaces ν-a.e. on Y. We denote the corresponding spectral measure on H and on H' by X and X', respectively. Similarly, $\mathcal{Q}_\varphi$ and $\mathcal{Q}'_\varphi$ denote the multiplication operators on H and on H', respectively. Corollary 1.8 will be generalized to more general situation as follows:

Theorem 2.3. Let μ, ν be σ-finite measure space (with a countable base) on a measurable space $(Y, \mathfrak{A})$.

(1) Assume that $\mu \sim \nu$, $\dim(\mathcal{H}(y)) = \dim(\mathcal{H}'(y))$ μ-a.e. on $y \in Y$ and there exists a measurable operator-valued function v defined μ-a.e. on Y which maps $\mathcal{H}(y)$ unitarily onto $\mathcal{H}'(y)$. Then, $V : h \mapsto \mathcal{Q}_p v h$ is a unitary operator from H onto H' such that $V\mathcal{Q}_\varphi = \mathcal{Q}'_\varphi V$, where $p = \left(\frac{d\mu}{d\nu}\right)^{\frac{1}{2}}$ is the square root of the Radon-Nikodym derivative of μ and ν. In particular, $VX(\delta) = X'(\delta)V$ for any $\delta \in \mathfrak{A}$.

(2) If V is an isometry from H onto H' such that $VX(\delta) = X'(\delta)V$ for any $\delta \in \mathfrak{A}$, then we have $\mu \sim \nu$, $\dim(\mathcal{H}(y)) = \dim(\mathcal{H}'(y))$ μ-a.e. on $y \in Y$, and V admits the representation $(Vh)(y) = p(y)v(y)h(y)$ for μ-a.e. on $y \in Y$.

Proof. Here, we omit this proof. See [BS, Chapter 7, Theorem 2.2]. □

Lemma 2.4. Let t be a measurable operator-valued function defined μ-a.e. on Y which satisfies $t(y) \in B(\mathcal{H}(y))$ for $y \in Y$ and $\mu\text{-}\sup \|t(y)\|_{B(\mathcal{H}(y))} < \infty$. Then, the operator $T : H \to H$ which is given by

$$(Th)(y) = t(y)h(y) \tag{1}$$

is a bounded operator which commutes with $\mathcal{Q}_\varphi$ for any measurable $\mathbb{C}$-valued function φ defined μ-a.e. on Y. In particular, T commutes with $X(\delta)$ for any $\delta \in \mathfrak{A}$, and

$$\|T\| = \mu\text{-}\sup \|t(y)\|_{B(\mathcal{H})}. \tag{2}$$

Proof. This follows by direct computations except to norm equality (2). It is easy to see that

$$\|T\| \leq \mu\text{-}\sup \|t(y)\|_{B(\mathcal{H}(y))}.$$

The opposite follows from the proof Theorem 2.6 (we omit it in this note. see [BS, Chapter 7, Theorem 2.3 (b)]). □

Definition 2.5. The operator $T : \int^{\oplus} \mathcal{H}(y)\, d\mu(y) \to \int^{\oplus} \mathcal{H}(y)\, d\mu(y)$ having the form (1) in Lemma 2.4 is called a *decomposable operator*, denoted by $T = \int_Y^{\oplus} t(y)\, d\mu(y)$.

The following is the main theorem in this note. This is a generalization of diagonalization:

Theorem 2.6. If $T \in B(H)$ commutes with $X(\delta)$ for any $\delta \in \mathfrak{A}$, then T admits the representation (1) in Lemma 2.4, that is, there exists a measurable operator-valued function t defined μ-a.e. on Y such that $t(y) \in B(\mathcal{H}(y))$, $\mu\text{-}\sup \|t(y)\|_{B(\mathcal{H}(y))} < \infty$ and

$$T = \int_Y^{\oplus} t(y)\, d\mu(y).$$

We only state the sketch of the proof of this theorem. To show this theorem, we have to construct some operator-valued function t defined on μ-a.e. on Y such that $t(y) \in B(\mathcal{H}(y))$. It is natural to define t by $t(y)h(y) := (Th)(y)$. But, there is a big difference between the continuity of $t(y)$ and that of T. The continuity of $T \in B(H)$ is related to the topology of H, that is, L^2-convergence. On the other hand, the continuity of $t(y)$ for some $y \in Y$ is related to the topology of $\mathcal{H}(y)$, that is, pointwise convergence. Thus, seeing the continuity of $t(y)$ from that of T means roughly seeing the pointwise convergence from L^2-convergence. For that, we need the sequence which approximate some L^2-function not only in L^2-topology but also in pointwise-topology ($\mathcal{H}(y)$-topology). However, in general, a sequence which L^2-converges does not pointwisely converge (remark that they have a subsequence which pointwisely converges). Thus, most important point of the proof is *to find a nice* $Y_0 \in \mathfrak{A}$ *such that* $\mu(Y \setminus Y_0) = 0$ *and some natural* L^2*-approximation sequence also pointwisely converges on* Y_0. For details, see [BS, Chapter 7, Theorem 2.3 (b)].

3. Decomposability of scattering operator

Let us recall some facts about scattering operator. Let $\mathcal{H}$ be a separable Hilbert space and $U \in B(\mathcal{H})$ be a unitary operator. A *core* for U is a Borel set $\widehat{\sigma} \subset [0, 2\pi)$ such that $E^U([0, 2\pi) \setminus \widehat{\sigma}) = 0$ ($\widehat{\sigma}$ supports the spectral measure of U) and if σ' is another Borel set which supports the spectral measure of U, then $|\widehat{\sigma} \setminus \sigma'| = 0$ (the left hand side is the Lebesgue measure of $\widehat{\sigma} \setminus \sigma'$). Then, we have the following fact by the spectral theorem:

Fact 3.1. For a.e. $\theta \in \widehat{\sigma}$, there exists a separable Hilbert space $\mathcal{H}(\theta)$ and a map $\mathcal{F} : \mathcal{H} \to \int_{\widehat{\sigma}}^{\oplus} \mathcal{H}(\theta)\, d\theta$ ($d\theta$ is the Lebesgue measure) such that $\mathcal{F}|_{\mathcal{H}_{\mathrm{s}}(U)} = 0$ and $\mathcal{F}_{\mathrm{ac}}$ is a unitary operator from $\mathcal{H}_{\mathrm{ac}}(U)$ to $\int_{\widehat{\sigma}}^{\oplus} \mathcal{H}(\theta)\, d\theta$, and

$$\mathcal{F}_{\mathrm{ac}} U_{\mathcal{H}_{\mathrm{ac}}(U)} \mathcal{F}_{\mathrm{ac}}^* = \int_{\widehat{\sigma}}^{\oplus} e^{i\theta}\, d\theta.$$

Note that the absolutely continuous part $E_{\mathrm{ac}}^U(\delta)$ of spectral measure of U corresponds to $X(\delta)$ in $\int_{\widehat{\sigma}}^{\oplus} \mathcal{H}(\theta)\, d\theta$. Let $\mathcal{H}_0$ be another separable Hilbert space and $U_0 \in B(\mathcal{H}_0)$ be a unitary operator. Let also $\mathcal{J}$ be an bounded operator from $\mathcal{H}_0$ to $\mathcal{H}$. The expressions $W_+(U, U_0, \mathcal{J}, \Theta)$ and $W_-(U, U_0, \mathcal{J}, \Theta)$ denote the wave operators for any Borel set $\Theta \subset [0, 2\pi)$. Then, the *scattering operator* is defined by

$$S(U, U_0, \mathcal{J}, \Theta) := W_+(U, U_0, \mathcal{J}, \Theta)^* W_-(U, U_0, \mathcal{J}, \Theta) \in B(\mathcal{H}_0).$$

By the intertwining property of wave operator, we have that $S(U, U_0, \mathcal{J}, \Theta)$ commutes with U_0. Then, applying Fact 3.1 to $(\mathcal{H}_0, U_0)$ and using Theorem 2.6, we have the following fact:

Corollary 3.2. The operator $S(U, U_0, \mathcal{J}, \Theta)$ is decomposable and

$$\mathcal{F}_{\mathrm{ac}} S(U, U_0, \mathcal{J}, \Theta)|_{\mathcal{H}_{\mathrm{ac}}(U_0)} \mathcal{F}_{\mathrm{ac}}^* = \int_{\widehat{\sigma}}^{\oplus} s(\theta)\, d\theta$$

with s a measurable operator-valued function defined $d\theta$-a.e. on $\widehat{\sigma}$ such that $s(\theta) \in B(\mathcal{H}(\theta))$ and $\|S(U, U_0, \mathcal{J}, \Theta)|_{\mathcal{H}_{\mathrm{ac}}(U_0)}\| = d\theta\text{-}\sup \|s(\theta)\|_{\mathcal{H}(\theta)}$.

References

[BS] M. S. Birman & M. Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Spaces*, D. Reidel Publishing Company.