Partial isometries and wave operators

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1. Some basics of operator

Definition 1.1. Let H and K be Hilbert spaces. We say that $W \in B(H,K)$ is a *partial isometry* if W is an isometry on $\ker(W)^{\perp}$.

Lemma 1.2. The range of any partial isometry $W \in B(H,K)$ is closed.

Proof. Assume that a sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ is such that $Wx_n \to y$ as $n \to \infty$ for some $y \in K$. Let $P \in B(H)$ be the orthogonal projection onto $\ker(W)^{\perp}$. Then, we have that

$$\|Px_n - Px_m\|_H = \|WPx_n - WPx_m\|_K = \|Wx_n - Wx_m\|_H \to 0,$$

that is, $\{Px_n\}$ is a Cauchy sequence (remark that $(\ker(W)^{\perp})^{\perp} = \ker(W)$). By the completeness of H, there exists $x \in \ker(W)^{\perp}$ such that $Px_n \to x$ (note that the orthogonal subspace of any subset is closed). Thus, we have that

$$\|Wx - y\|_K = \lim_n \|Wx - Wx_n\|_K = \|Wx - WPx_n\|_K = \|x - Px_n\|_H \to 0,$$

that is, $Wx = y$. □

Lemma 1.3. Let H and K be Hilbert spaces, and M and N be closed subspaces of H and K, respectively. For $A \in B(H,K)$ such that $AM \subset N$, it holds that $A^*N^{\perp} \subset M^{\perp}$.

Proof. Assume that $AM \subset N$. For any $x \in N^{\perp}$ and any $y \in M$, we have that

$$\langle A^*x, y\rangle = \langle x, Ay\rangle = 0$$

since $Ay \in N$. Thus, the statement holds. □

2. Wave operator

Let us recall the definition of the wave operator.

Definition 2.1. Let H and H_0 be Hilbert spaces. Let $U \in B(H)$ and $U_0 \in B(H_0)$ be unitary operators on each space. Also, let J be a bounded operator from H_0 to H, which is called *identification operator*. If there exists a SOT-limit $\text{s-lim}_n U^{-n}JU_0^n$, then we call it the *wave operator* and denote it by $W_{\pm} = W_{\pm}(U, U_0, J)$. Moreover, for any Borel set $\Theta \subset [0, 2\pi)$, if there exists a SOT-limit $\text{s-lim}_n U^{-n}JU_0^{-n}E^{U_0}(\Theta)$, then we call it the *(local) wave operator*, where $E^{U_0}(\cdot)$ is the spectral measure of U_0. We denote it by $W_{\pm}(\Theta) = W_{\pm}(U, U_0, J, \Theta)$.

Lemma 2.2. If J is unitary, then $\ker(W_{\pm}(\Theta))$ is exactly $E^{U_0}([0, 2\pi) \setminus \Theta)$.

Proof. It is clear that $E^{U_0}([0, 2\pi) \setminus \Theta) \subset \ker(W_{\pm}(\Theta))$. On the other hand, for any $f \in H_0$ such that $W_{\pm}(\Theta)f = 0$, we have that

$$\|E^{U_0}(\Theta)f\|_{H_0} = \|U^{-n}JU_0^nE^{U_0}(\Theta)f\|_H \to 0.$$

This implies that $f \in \text{ran}(E^{U_0}([0, 2\pi) \setminus \Theta))$. □

Proposition 2.3. If J is unitary, the wave operator $W_{\pm}(\Theta)$ is a partial isometry and its range is closed.

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Proof. By Lemma 1.2, it suffices to show that $W_\pm(\Theta)$ is a partial isometry. For any $f \in H_0$, we have that

$$\|W_\pm(\Theta)E^{U_0}(\Theta)f\|_H = \lim_n \|U^{-n}JU_0^n E^{U_0}(\Theta)f\|_H = \lim_n \|E^{U_0}(\Theta)f\|_{H_0} = \|E^{U_0}(\Theta)f\|_{H_0}.$$

Combining this with Lemma 2.2, we have that $W_\pm(\Theta)$ is a partial isometry. □

Definition 2.4. We define the subspace $\mathcal{N}_\pm(\Theta)$ as

$$\left\{ f \in H \mid \lim_n \|J^* U^n E^U(\Theta)f\|_{H_0} = 0 \right\}.$$

Lemma 2.5. The $\mathcal{N}_\pm(\Theta)$ is closed.

Proof. Let $\{f_m\} \subset \mathcal{N}_\pm(\Theta)$ be a sequence such that $f_m \to f$ as $m \to \infty$ for some $f \in H$. Then, for any $\epsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\|f - f_{m_0}\| < \epsilon$. Also, for $m_0 \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $\|J^* U^{n_0} E^U(\Theta) f_{m_0}\| < \epsilon$. Hence, we have that

$$\begin{aligned} \|J^* U^{n_0} E^U(\Theta) f\| &\le \|J^* U^{n_0} E^U(\Theta) f_{m_0}\| + \|J^* U^{n_0} E^U(\Theta)(f - f_{m_0})\| \\ &\le \|J^* U^{n_0} E^U(\Theta) f_{m_0}\| + \|J\|\epsilon \\ &\le (1 + \|J\|)\epsilon, \end{aligned}$$

that is, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|J^* U^{n_0} E^U(\Theta) f\| \le (1 + \|J\|)\epsilon. \tag{1}$$

This implies that $f \in \mathcal{N}_\pm(\Theta)$. □

Proposition 2.6. The U is reduced by $\mathcal{N}_\pm(\Theta)$.

Proof. Consider the decomposition $H = \mathcal{N}_\pm(\Theta) \oplus \mathcal{N}_\pm(\Theta)^\perp$. The representation operator matrix of U with respect to this decomposition is

$$\begin{bmatrix} PU|_{\mathcal{N}_\pm(\Theta)} & PU|_{\mathcal{N}_\pm(\Theta)^\perp} \\ (I-P)U|_{\mathcal{N}_\pm(\Theta)} & (I-P)U|_{\mathcal{N}_\pm(\Theta)^\perp} \end{bmatrix},$$

where P is the orthogonal projection onto $\mathcal{N}_\pm(\Theta)$. Here, remark that $U\mathcal{N}_\pm(\Theta) \subset \mathcal{N}_\pm(\Theta)$. In fact, for any $f \in \mathcal{N}_\pm(\Theta)$, we have that

$$\|J^* U^n E^U(\Theta) U f\|_{H_0} = \|J^* U^{n+1} E^U(\Theta) f\|_{H_0} \to 0.$$

Similarly, we can also see that $U^*\mathcal{N}_\pm(\Theta) \subset \mathcal{N}_\pm(\Theta)$. Also, by Lemma 1.3, it implies that $U\mathcal{N}_\pm(\Theta)^\perp \subset \mathcal{N}_\pm(\Theta)^\perp$. Thus, we have that $PU|_{\mathcal{N}_\pm(\Theta)^\perp} = 0$ and $(I-P)U|_{\mathcal{N}_\pm(\Theta)} = 0$, that is,

$$U = \begin{bmatrix} PU|_{\mathcal{N}_\pm(\Theta)} & 0 \\ 0 & (I-P)U|_{\mathcal{N}_\pm(\Theta)^\perp} \end{bmatrix}.$$

□

See e.g. [H] for more information about partial isometries.

References

[H] P. R. Halmos, *A Hilbert Space Problem Book, second edition*, Springer, 1982.