Proof of Woodbury Matrix Identity and the Kalman Update Formula

The statement of the **Woodbury Matrix Identity** is:

> For $E \in M_{n\times n}(\mathbb{R})$ and $G \in M_{m\times m}(\mathbb{R})$ both invertible, $F \in M_{n\times m}(\mathbb{R})$, and $H \in M_{m\times n}(\mathbb{R})$,
> $$(E + FGH)^{-1} = E^{-1} - E^{-1}F(G^{-1} + HE^{-1}F)^{-1}HE^{-1}.$$

The simplest proof is the **direct proof**, in which we just multiply $(E + FGH)$ with its alleged inverse:

$$\begin{aligned}
&(E + FGH)(E + FGH)^{-1}\\
&= (E + FGH)(E^{-1} - E^{-1}F(G^{-1} + HE^{-1}F)^{-1}HE^{-1})\\
&= EE^{-1} - EE^{-1}F(G^{-1} + HE^{-1}F)^{-1}HE^{-1} + FGHE^{-1} - FGHE^{-1}F(G^{-1} + HE^{-1}F)^{-1}HE^{-1}\\
&= (I + FGHE^{-1}) - \big(F(G^{-1} + HE^{-1}F)^{-1}HE^{-1} + FGHE^{-1}F(G^{-1} + HE^{-1}F)^{-1}HE^{-1}\big)\\
&= I + FGHE^{-1} - \big(F + FGHE^{-1}F\big)\big(G^{-1} + HE^{-1}F\big)^{-1}HE^{-1}\\
&= I + FGHE^{-1} - FG\big(G^{-1} + HE^{-1}F\big)\big(G^{-1} + HE^{-1}F\big)^{-1}HE^{-1}\\
&= I + FGHE^{-1} - FGIHE^{-1} = I + FGHE^{-1} - FGHE^{-1} = I.
\end{aligned}$$

Here, I denotes the identity matrix.

However, this proof is *lame and uninspired*. Let us introduce a more interesting way to prove the statement.

In this method, there are a few useful identities that we will use to prove the Woodbury Identity.
The first required identity is:

> For I the identity matrix, U and V *conformable* (chosen such that the operations required are defined),
> $$(I + UV)^{-1}U = U(1 + VU)^{-1}. \qquad (1)$$

The proof of this identity is quite simple. We consider

$$\begin{aligned}
U(I + VU) = (I + UV)U &\Longrightarrow (I + UV)^{-1}U(I + VU)(I + VU)^{-1} = (I + UV)^{-1}(I + UV)U(I + VU)^{-1}\\
&\Longrightarrow (I + UV)^{-1}UI = IU(I + VU)^{-1} \Longrightarrow (I + UV)^{-1}U = U(I + VU)^{-1}.
\end{aligned}$$

The second identity is

> For E, F, G and H defined in the Woodbury Matrix Identity,
> $$(E + FGH)^{-1}FG = E^{-1}F(G^{-1} + HE^{-1}F)^{-1}. \qquad (2)$$

The proof of this identity is as follows. First, by property of the inverse of a matrix $(AB)^{-1} = B^{-1}A^{-1}$,

$$(E + FGH)^{-1}FG = E^{-1}\big(I + FGHE^{-1}\big)^{-1}FG.$$

Set $U = F$ and $V = GHE^{-1}$. By the result of (1) we obtain:

$$(E + FGH)^{-1}FG = E^{-1}F\big(I + GHE^{-1}F\big)^{-1}G.$$

Again, by the property of inverses of matrices, namely $B^{-1}A^{-1} = (AB)^{-1}$,

$$\begin{aligned}
(E + FGH)^{-1}FG &= E^{-1}F\big(G^{-1} + G^{-1}GHE^{-1}F\big)^{-1}\\
&= E^{-1}F(G^{-1} + HE^{-1}F)^{-1}.
\end{aligned}$$

This proves the two identities that we need. Next, we prove the Woodbury Matrix Identity:

$$\begin{aligned} E^{-1} &= (E+FGH)^{-1}(E+FGH)E^{-1} = (E+FGH)^{-1}(I+FGHE^{-1}) \\ &= (E+FGH)^{-1} + (E+FGH)^{-1}FGHE^{-1} \\ &= (E+FGH)^{-1} + E^{-1}F(G^{-1}+HE^{-1}F)^{-1}HE^{-1}\,. \end{aligned}$$

Rearranging gives the statement of the Woodbury Matrix Identity, thereby proving it:

$$(E+FGH)^{-1} = E^{-1} - E^{-1}F\left(G^{-1}+HE^{-1}F\right)^{-1}HE^{-1}\,.$$

Next, we will use this identity to derive the **Kalman Update Formula**.

In the lecture, we defined P^a and $\bar{x}^a$ as:

$$\begin{aligned} P^a &:= \left(P^{-1}+H^TR^{-1}H\right)^{-1}\,, \\ \bar{x}^a &:= \bar{x} - P^aH^TR^{-1}\left(H\bar{x}-y_{obs}\right)\,. \end{aligned}$$

Before all else, we should check that we have no problem of dimensions in the two equations above. Here, $P \in M_{n\times n}(\mathbb{R})$, $H \in M_{m\times n}(\mathbb{R})$, $R \in M_{m\times m}(\mathbb{R})$, $\bar{x} \in M_{n\times 1}(\mathbb{R})$ and $y_{obs} \in M_{m\times 1}(\mathbb{R})$. Checking dimensions we see that there is no problem in dimensions, and we obtain that $P^a \in M_{n\times n}(\mathbb{R})$ and $\bar{x}^a \in M_{n\times 1}$.

We now apply the Woodbury Matrix Identity to the equation for P^a to obtain:

$$\begin{aligned} P^a &= \left(P^{-1}\right)^{-1} - \left(P^{-1}\right)^{-1}H^T\left(\left(R^{-1}\right)^{-1} + H\left(P^{-1}\right)^{-1}H^T\right)^{-1}H\left(P^{-1}\right)^{-1} \\ &= P - PH^T\left(R+HPH^T\right)^{-1}HP\,. \end{aligned}$$

Next, for $\bar{x}^a$, we first find an expression for $P^aH^TR^{-1}$:

$$\begin{aligned} P^aH^TR^{-1} &= PH^TR^{-1} - PH^T\left(R+HPH^T\right)^{-1}HPH^TR^{-1} \\ &= PH^T\left(R^{-1} - \left(R+HPH^T\right)^{-1}HPH^TR^{-1}\right)\,. \end{aligned}$$

We use the reverse Woodbury Matrix Identity, with $E=F=R$, $G=R^{-1}$, and $H=HPH^T$:

$$P^aH^TR^{-1} = PH^T\left(R+RR^{-1}HPH^T\right)^{-1} = PH^T\left(R+HPH^T\right)^{-1}\,.$$

We then find an expression for $\bar{x}^a$ using the expression above:

$$\bar{x}^a = \bar{x} - PH^T\left(R+HPH^T\right)^{-1}\left(H\bar{x}-y_{obs}\right)\,.$$

With this, we have derived the **Kalman Update Formula**. Its statement is:

For a system with P, H, R, $\bar{x}$ and y_{obs} as defined above, P^a and $\bar{x}^a$ are given by:

$$\begin{aligned} P^a &= P - PH^T\left(HPH^T+R\right)^{-1}HP\,, \\ \bar{x}^a &= \bar{x} - PH^T\left(HPH^T+R\right)^{-1}\left(H\bar{x}-y_{obs}\right)\,. \end{aligned}$$