Extension of the definition of an affiliated observable

M1 Ayumi UKAI (Student ID:322201078)

Let H be a self-adjoint operator on a Hilbert space $\mathbf{H}$ affiliated to a C^*-subalgebla $\mathcal{C} \subseteq \mathcal{B}(\mathbf{H})$, that is, $(H - z_0)^{-1} \in \mathcal{C}$ for some $z_0 \in \mathbb{C}\backslash\mathbb{R}$. Let us show that H induces $*$-homomorphism which is an observable affiliated to $\mathcal{C}$. Indeed, let Φ be the map $f \mapsto f(H)$ for any $f \in C_0(\mathbb{R})$. We show that Φ satisfies the following propositions:

(i) Φ is a $*$-homomorphism from $C_0(\mathbb{R})$ to $\mathcal{C}$.
(ii) $\sigma(H) = \sigma(\Phi)$.

As for (i), the only statement to show is that $\mathrm{Ran}(\Phi)$, the range of Φ, is contained in $\mathcal{C}$. From the definition of functional calculus, it is clear that Φ is a $*$-homomorphism. To show that $\mathrm{Ran}(\Phi) \subseteq \mathcal{C}$, we only need to show that $(H - z)^{-1} \in \mathcal{C}$ for any $z \in \mathbb{C}\backslash\mathbb{R}$. This is a consequence of Stone-Weierstrass theorem. That is, $\mathcal{F} := \{f \in C_0(\mathbb{R}) | \Phi(f) \in \mathcal{C}\}$ becomes a sub C^*-algebra of $C_0(\mathbb{R})$. Suppose that $g_z(x) := (x - z)^{-1} \in \mathcal{C}$ for any $z \in \mathbb{C}\backslash\mathbb{R}$. The family of functions $\{g_z\}$ separates any points of $\mathbb{R}$. Thus, Stone-Weierstrass theorem tells us that $\mathcal{F} = C_0(\mathbb{R})$ and this is just the statement that $\mathrm{Ran}(\Phi) \subseteq \mathcal{C}$.

Let us show that $(H - z)^{-1} \in \mathcal{C}$ for any $z \in \mathbb{C}\backslash\mathbb{R}$. Let $\mathcal{R} := \{z \in \mathbb{C}\backslash\mathbb{R} | (H - z)^{-1} \in \mathcal{C}\}$. Since H is affiliated to $\mathcal{C}$, there exists some $z_0 \in \mathcal{R}$. For any $w \in \mathcal{R}$, as long as $z \in \mathbb{C}\backslash\mathbb{R}$ satisfies $|z - w| \|(H - w)^{-1}\| < 1$, the Neumann series leads the following equation:

$$(H - z)^{-1} = (H - w)^{-1}(1 + (z - w)(H - w)^{-1})^{-1} = \sum_{n=0}^{\infty} (z - w)^n ((H - w)^{-1})^{n+1}. \tag{1}$$

Since each term in the series belongs to $\mathcal{C}$, $(H - z)^{-1} \in \mathcal{C}$. The remainder issue is, then, that for any $z \in \mathbb{C}\backslash\mathbb{R}$ there exists some $w \in \mathcal{R}$. Indeed, observe firstly that the following inequality holds:

$$\|(H - w)^{-1}\| = \sup_{\lambda \in \sigma(H)} \{|\lambda - (\Re w + i\Im w)|^{-1}\} \leq \frac{1}{|\Im w|}.$$

This implies that if $|z - w| < |\Im w|$ then $z \in \mathcal{R}$. Then one observes that $\bar{z_0} \in \mathcal{R}$, and that $z \in \mathcal{R}$ for every z whose real part is equal to $\Re z_0$. (See Fig.1 on the next page.) Furthermore, the following inequality also holds:

$$|z - w|^2 \|(H - w)^{-1}\|^2 \leq \frac{(\Re z - \Re w)^2 + (\Im z - \Im w)^2}{\Im w^2} = 1 - \frac{2\Im z}{\Im w} + \frac{1}{\Im w^2}((\Re z - \Re w)^2 + \Im z^2).$$

Thus, when $\Im z > 0$, $|z - w| \|(H - w)^{-1}\| < 1$ for some $\Im w$ large enough and when $\Im z < 0$, for some $-\Im w$ large enough. And $w \in \mathcal{R}$ with $\Im w$ or $-\Im w$ large enough can be taken among elements whose real part is equal to $\Re z_0$. Therefore, for any $z \in \mathbb{C}\backslash\mathbb{R}$ there exists some $w \in \mathcal{C}$ such that (1) holds and $(H - z)^{-1} \in \mathcal{R}$. (i) has, then, been shown.

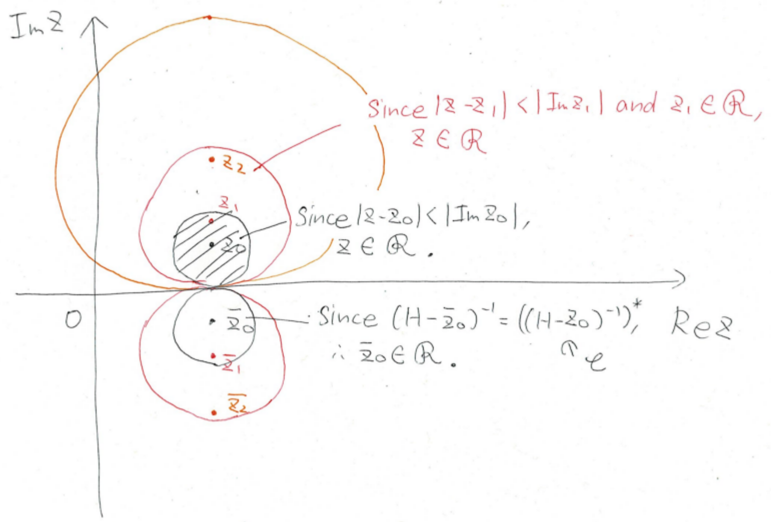

Fig.1

As for (ii), this is immediately derived from the next lemma.

Lemma 0.1. *For any self-adjoint operator, its spectrum and spectral support are equal. In other words, for any self-adjoint operator H and for any $\lambda \in \mathbb{R}$, $\lambda \in \sigma(H)$ if and only if $\mu^H(I(\lambda, \epsilon)) \neq 0$ for any $\epsilon > 0$, where μ^H denotes the spectral measure of H and $I(\lambda, \epsilon)$ denotes the interval $(\lambda - \epsilon, \lambda + \epsilon) \subseteq \mathbb{R}$*

For its proof, see the students report "*spectrum and spectral support*" by Huyga Ito.

For any $\lambda \notin \sigma(H)$, by the lemma just mentioned, there exists some $\epsilon > 0$ such that $\mu^H(I(\lambda, \epsilon)) = 0$. Using Urysohn's lemma, there exists a function $g \in C_0(\mathbb{R})$, which takes 1 on $\lambda \in \mathbb{R}$ and takes 0 anywhere but $I(\lambda, \epsilon)$. $\Phi(g) = g(H) = 0$ and this implies $\lambda \notin \sigma(\Phi)$. On the other hand, suppose that $\lambda \in \sigma(H)$. For any functions $\varphi \in C_0(\mathbb{R})$ which satisfies $\varphi(\lambda) \neq 0$, $e^{-i\theta}\varphi(\lambda) := (\varphi(\lambda)/|\varphi(\lambda)|)^{-1}\varphi(\lambda) > 0$ and there exists some $\epsilon > 0$ such that $e^{-i\theta}\varphi > 0$ on $I(\lambda, \epsilon)$. From the assumption and the above lemma, $\mu^H(I(\lambda, \epsilon)) \neq 0$ and $e^{-i\theta}\varphi(H) \neq 0$. Thus, $\varphi(H) \neq 0$ and $\lambda \in \sigma(\Phi)$.