Summary : random variable

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space : Ω = set of all possible events, $\mathcal{F}$ = family of subsets of Ω + some stability properties, $\mathbb{P}$ = a function giving a weight to all element of $\mathcal{F}$, with $\mathbb{P}(\Omega) = 1$.
- $(\Lambda, \mathcal{E})$ a measurable space : Λ = a set, $\mathcal{E}$ = family of subsets of Λ + some stability properties. Examples : $(\mathbb{R}^N, \mathcal{T}_B)$ family of subsets of $\mathbb{R}^N$ generated by boxes, or $(\{\lambda_1, \ldots, \lambda_N\}, P(\{\lambda_1, \ldots, \lambda_N\}))$ a finite set and its power set = family of all possible subsets (2^N subsets)
- X a random variable : function from Ω to Λ satisfying $X^{-1}(A) \equiv \{\omega \in \Omega \mid X(\omega) \in A\} \in \mathcal{F}$ $\forall A \in \mathcal{E}$. X should be interpreted as a "question" on the complicated set Ω, and Λ is the set of possible answers.

much simpler than Ω

• μ_X the <u>induced probability measure</u>, or <u>law of X</u> : function giving a weight to all elements of $\mathcal{E}$ by

$$\mu_X(A) := \mathbb{P}(X \in A) = \underbrace{\mathbb{P}(\underbrace{\{\omega \in \Omega \mid X(\omega) \in A\}}_{\in \mathcal{F}})}_{\in [0,1]},$$

for any $A \in \mathcal{E}$.

• <u>The expectation</u> : for $(\Xi, \mathcal{G})$ a new measurable space (typically $\Xi = \mathbb{R}$, or $\mathbb{R}^N$, or $M_{n\times m}(\mathbb{R})$) and for $f : \Lambda \to \Xi$ satisfying $f^{-1}(B) = \{x \in \Lambda \mid f(x) \in B\} \in \mathcal{E}$ $\forall\, B \in \mathcal{G}$, the <u>expectation of $f(X)$</u> is defined by

$$\mathbb{E}(f(X)) = \underbrace{\int_\Lambda f(x)\, \mu_X(dx)}_{= \text{ notation}} = \lim_j \sum_j f(x_j)\, \mu_X(f^{-1}(B_j))$$

Lebesgue type integral.

• 2 Principal types of random variables :

1) $(\Lambda, \mathcal{E}) = (\mathbb{R}^N, \mathcal{T}_B)$ and $\int_\Lambda f(x)\, \mu_X(dx) = \overbrace{\int_{\mathbb{R}^N} f(x)\, \Pi_X(x)\, dx}^{\text{usual } N\text{-dim integral}}$

absolutely continuous

with $\Pi_X : \mathbb{R}^N \to [0,\infty)$ satisfying $\int_{\mathbb{R}^N} \Pi_X(x)\, dx = 1$.

$\uparrow$ pdf

2) $(\Lambda, \mathcal{E}) = (\{\text{countable set}\}, \text{power set})$ and $\int_\Lambda f(x)\, \mu_X(dx) =$

discrete

$= \sum_x f(x)\, p_X(x)$ with $p_X(x) := \mathbb{P}(X^{-1}(\{x\}))$.

$\uparrow$ pmf