C^*-ALGEBRAIC METHODS IN SPECTRAL THEORY

Various exercises from the lecture notes

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Exercise 1.5.4 Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence with $f_n \in D(A)$ which strongly converges to $f \in \mathcal{H}$. We show first that $\{Af_n\}_{n\in\mathbb{N}}$ strongly converges, that is, A is continuous. Let $\epsilon > 0$ and there exists an integer N s.t. if $m > N$, then $||lim_n\ f_n - f_m|| < \frac{\epsilon}{c}$. Then $||lim_n\ Af_n - Af_m|| = lim_n||A(f_n - f_m)|| \leq lim_n\ c||f_n - f_m|| = c||lim_n\ f_n - f_m|| < \epsilon$. Here, if $\{f'_n\}_{n\in\mathbb{N}}$ is another sequence in $D(A)$ which converges to f, then $\{Af'_n\}_{n\in\mathbb{N}}$ converges to the same element as the one to which $\{Af_n\}_{n\in\mathbb{N}}$ converges. To see this, let N be a natural number such that $||f_n - f|| < \frac{\epsilon}{2c}, ||f'_m - f|| < \frac{\epsilon}{2c}$ for $m, n > N$. We get $||f_n - f'_m|| < ||f_n - f|| + ||f'_m - f|| < \frac{\epsilon}{c}$. This leads to $||Af_n - Af'_m|| \leq c||f_n - f'_m|| < \epsilon$, concluding that $||lim_n Af_n - Af'_m|| < \epsilon$ as desired.

Let $f \in \mathcal{H}\backslash D(A)$. Since $D(A)$ is dense, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of $D(A)$ strongly converging to f. We define $A\overline{f} = lim_n\ Af_n$ (note that it is independent on choice of $\{f_n\}_{n\in\mathbb{N}}$ due to continuity of A). Let $\epsilon > 0$. There exists an integer N s.t. for all $n > N$, we have $||Af - Af_n|| \overline{<} \epsilon/2$ and $||f - f_n|| < \epsilon/2c$. We can estimate $||Af||$as follows; $\overline{}$

$$
\begin{aligned}
||\overline{A}f|| &\leq ||Af_n|| + ||\overline{A}f - Af_n|| \\
&< ||Af_n|| + \frac{\epsilon}{2} \\
&\leq c||f_n|| + \frac{\epsilon}{2} \\
&< c||f|| + \epsilon
\end{aligned}
$$

where we use triangle inequality in the first and the last inequality (as for the latter, we use $||f_n|| - ||f|| \leq ||f - f_n|| < \epsilon/2c$). Taking $\epsilon \to +0$, we obtain $||\overline{A}f|| \leq c||f||$.

Exercise 1.3.3

$$\begin{aligned}||A_n|| &= \sup_{f\in\mathcal{H}} \frac{||A_n f||}{||f||} \\ &= \sup_{f\in\mathcal{H}} \frac{||\sum_j \langle f, g_j\rangle h_j||}{||f||} \\ &\leq \sup_{f\in\mathcal{H}} \frac{\sum_j |\langle f, g_j\rangle| \cdot ||h_j||}{||f||} \\ &= \sum_j ||\varphi_{g_j}|| \cdot ||h_j|| \\ &= \sum_j ||g_j|| \cdot ||h_j||\end{aligned}$$

Moreover we have

$$\begin{aligned}\langle A_n f, f'\rangle &= \langle \sum_j \langle f, g_j\rangle h_j, f'\rangle \\ &= \sum_j \langle f, g_j\rangle \langle h_j, f'\rangle \\ &= \sum_j \langle f, \overline{\langle h_j, f'\rangle} g_j\rangle \\ &= \langle f, \sum_j \overline{\langle h_j, f'\rangle} g_j\rangle \\ &= \langle f, \sum_j \langle f', h_j\rangle g_j\rangle.\end{aligned}$$

So we get $A_n^* f = \sum_j \langle f, h_j\rangle g_j$.

Exercise 1.6.5 For $f' \in D(A)$ and $z, z' \in \mathbb{C}$, $(A-z)f' \in D(A)$ iff $(A-z')f' \in D(A)$ as the difference $(z-z')f'$ is contained in $D(A)$. So the operators $(A-z)(A-z')$ and $(A-z')(A-z)$ have the same domain D. Note that if $f' \in D$, then $Af' \in D(A)$, by which we can compute expansions of $(A-z)(A-z')f'$ and $(A-z')(A-z)f'$ for $f' \in D$, concluding that $(A-z)(A-z') = (A-z')(A-z)$. Since $z_1, z_2 \in \rho(A)$, for any $f \in \mathcal{H}$, there exists $g \in D$ such that $f = (A-z_1)(A-z_2)g = (A-z_2)(A-z_1)g$. We have $((A-z_1)^{-1} - (A-z_2)^{-1})f = ((A-z_2)-(A-z_1))g = (z_1-z_2)g$ while $(z_1-z_2)(A-z_1)^{-1}(A-z_2)^{-1}f = (z_1-z_2)g$. As f is arbitrary, we get $(A-z_1)^{-1} - (A-z_2)^{-1} = (z_1-z_2)(A-z_1)^{-1}(A-z_2)^{-1}$.

Exercise 3.3.7 First, we show that $(C_0(G), G, L)$ is a C^*-dynamical system. We need to show that $x \mapsto L_x f$ is continuous for each $f \in C_0(G)$, that is, for any $\epsilon > 0$, there exists an open neighborhood V of x such that for all $y \in V$, $sup_{z\in G}|f(y^{-1}z) - f(x^{-1}z)| < \epsilon$. By replacing $x^{-1}z$ by z, we can assume that $x = e$ is the identity element of G. Since $f \in C_0(G)$, there exists a compact subset $K' \subset G$ with the property that if $z \in G\backslash K'$, then $|f(z)| < \frac{\epsilon}{2}$. As G is locally compact, there exists a compact neighborhood K'' of e. Let $K = K'' \cdot K'$. This is a compact subset since the multiplication $G \times G \to G$ is continuous and the image of a compact set (in our case, $K'' \times K'$) under a continuous map is compact. If $K = G$, the necessary argument is given in the following paragraph. Assume that $K \neq G$. For fixed $y \in K''$, if $z \in G \backslash (yK' \cup K')$, then $|f(y^{-1}z) - f(z)| < |f(y^{-1}z)| + |f(z)| < \epsilon$. Varying y in K'', we see that if $z \in G\backslash K$ and $y \in K''$, then $|f(y^{-1}z) - f(z)| < \epsilon$. It suffices to find an open neighborhood U of e such that if $y \in U$, then $|f(y^{-1}z) - f(z)| < \epsilon$ for all $z \in K$. Given such U, $V = U \cap K''^{\circ}$ has the desired property where K''° denotes for the interior of K''.

We claim that for each $z \in G$ and any $\epsilon > 0$, there exists an open neighborhood U_z of e such that if $a, b \in U_z$, then $|f(az) - f(bz)| < \epsilon$. Indeed, as f and the multiplication with z are continuous, there exists an open neighborhood $U_z \ni e$ with the property that if $c \in U_z$, then $|f(cz) - f(z)| < \frac{\epsilon}{2}$. By the triangle inequality, we get $|f(az) - f(bz)| \leq |f(az) - f(z)| + |f(bz) - f(z)| < \epsilon$ for $a, b \in U_z$.

If $y^{-1}g, g \in U_z$, then $|f(y^{-1}gz) - f(gz)| < \epsilon$. Since the multiplication in G is continuous, there exist two open neighborhoods U'_z, U''_z of e with $U'_z U''_z \subset U_z$. So we conclude that for fixed $z \in G$, there exist an open neighborhood $V_z (= U''_z z)$ of z and an open neighborhood $W_z (= U'_z)$ of e satisfying that for any points $z' \in V_z$ and $y^{-1} \in W_z$, $|f(y^{-1}z') - f(z')| < \epsilon$. Because K is compact, one can take a finite number of points $\{z_k\}_{1 \leq k \leq n}$ such that $\cup_k V_k \supset K$ where $V_k = V_{z_k}$. Let $W' = \cap_k W_k$ and set $W = W'^{-1}$ where $W_k = W_{z_k}$. Then W is an open subset (as the map associating to $g \in G$ its inverse is continuous) which satisfies the desired condition, i.e. $y \in W \Rightarrow |f(y^{-1}z) - f(z)| < \epsilon \; for \; all \; z \in K$.

Next, we show $(L^2(G), Id, U)$ is a covariant representation of $(C_0(G), G, L)$. It is known that $(L^2(G), U)$ is a representation of G. We will show $(L^2(G), Id)$ is a representation of $C_0(G)$. We first show that $Id(h)$ is an element of $\mathcal{B}(\mathcal{H})$ for $h \in C_0(G)$. We need to show $||Id(h)|| = \left(\frac{\int_G |Id(h)f|^2 d\mu}{\int_G |f|^2 d\mu} \right)^{1/2} < \infty$. For any $\epsilon > 0$, there exists a compact subset $K_\epsilon \subset G$ with the property that $g \in G \backslash K_\epsilon \Rightarrow h(g) < \epsilon$. Since a continuous function (whose value is in $\mathbb{R}$) defined over a compact set has a maximum value, there is $M > 0$ such that $|h(x)| \leq M$ for all $x \in K_\epsilon$. So we can estimate the numerator as follows; $\int_G |Id(h)f|^2 d\mu = \int_{K_\epsilon} |Id(h)f|^2 d\mu + \int_{G\backslash K_\epsilon} |Id(h)f|^2 d\mu < M^2 \int_{K_\epsilon} |f|^2 d\mu + \epsilon^2 \int_{G\backslash K_\epsilon} |f|^2 d\mu \leq max\{\epsilon^2, M^2\} \int_G |f|^2 d\mu$, which means that $||Id(h)|| < max\{\epsilon, M\}$. It is obvious that Id is a homomorphism. By the following computation, we see that it preserves involution, so it is actually a $*$-homomorphism; $\langle f, Id(h)g \rangle = \int f \overline{hg} d\mu = \int \overline{h} f \overline{g} d\mu = \langle \overline{h} f, g \rangle = \langle Id(\overline{h}) f, g \rangle$ for $h \in C_0(G)$ and $f, g \in L^2(G)$.

Finally, we need to show the equality $Id(L_x h) = U_x Id(h) U_x^*$ for any $h \in C_0(G)$. For $f \in L^2(G)$, we have $[Id(L_x h)f](y) = (L_x h(y))f(y)$ while $U_x Id(h) U_x^* f(y) = U_x Id(h)(f(xy)) = U_x(h(y)f(xy)) = h(x^{-1}y)f(y) = (L_x h(y))f(y)$. As y is arbitrary, the equality holds.

Exercise 4.1.3 For $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$[X_j, X_k]f = (iX_jX_k - iX_kX_j)f = (ix_jx_k - ix_kx_j)f = 0$$

$$[D_j, D_k]f = [D_jD_k - D_kD_j]f = [-\partial_j\partial_k - \partial_k\partial_j]f = 0$$

since the partial differentials commute on f which is a C^∞ function. Moreover,

$$\begin{aligned} [iD_j, X_k]f &= (iD_jX_k - X_k \cdot (iD_j))f \\ &= (\partial_j X_k - X_k\partial_j)f \\ &= \partial_j(x_k f) - x_k\partial_j f \\ &= \delta_{jk} f + x_k\partial_j f - x_k\partial_j f \\ &= \delta_{jk} f. \end{aligned}$$