report

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1. (Example of an unbounded operator)

Let $H = L^2(\mathbb{R}), D(X) = \{f \in H; \int_{\mathbb{R}} x^2|f(x)|^2\,dx < \infty\}$ where X acts on $D(X)$ by $[Xf](x) = xf(x)$ for all $x \in \mathbb{R}$ and $f_n(x) = \chi_{(n,n+1)}(x)$. Then for each $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} x^2|\chi_{(n,n+1)}(x)|^2\,dx = \int_n^{n+1} x^2\,dx = n^2 + n + \frac{1}{3} < \infty.$$

So, f_n is in the domain of X. Since $\|f_n\|_2 = 1$,

$$\|Xf_n\|_2^2 = \left(n^2 + n + \frac{1}{3}\right)\|f_n\|_2^2 \geq n^2\|f_n\|_2^2.$$

Therefore $\|Xf_n\|_2 \geq n\|f_n\|_2$.

Density of $D(X)$ in $L^2(\mathbb{R})$:

Let $f_n(x) = \chi_{(-n,n)}(x)f(x)$ for all $f \in L^2(\mathbb{R})$. Then for each $n \in \mathbb{N}$, we have

$$\int_{\mathbb{R}} x^2|f_n(x)|^2\,dx = \int_{-n}^{n} x^2|f(x)|^2\,dx \leq n^2\int_{\mathbb{R}}|f(x)|^2\,dx < \infty.$$

So, f_n is in the domain of X and since

$$\|f_n - f\|_2^2 = \int_{\mathbb{R}} |\chi_{(-n,n)}(x) - 1|^2|f(x)|^2\,dx,$$

and $|\chi_{(-n,n)}(x) - 1|^2|f(x)|^2 \leq 4|f(x)|^2$, applying the Lebesgue convergence theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}} |\chi_{(-n,n)}(x) - 1|^2|f(x)|^2\,dx = \int_{\mathbb{R}}\lim_{n\to\infty}|\chi_{(-n,n)}(x) - 1|^2|f(x)|^2\,dx = 0$$

Therefore $\overline{D(X)} = L^2(\mathbb{R})$.

2. (Self adjoint operators are closed)

Let $(A, D(A))$ be a self adjoint operator, i.e. $D(A) = D(A^*)$ and $Af = A^*f$ for all $f \in D(A)$. Now we take a sequence $\{f_n\}_{n=1}^{\infty} \subset D(A)$ such that $s\text{-}\lim_{n\to\infty} f_n = f \in H$ and there exists $h := s\text{-}\lim_{n\to\infty} Af_n$. Then for all $g \in D(A)$, we have

$$\begin{aligned}\langle f, Ag\rangle &= \lim_{n\to\infty}\langle f_n, Ag\rangle \\ &= \lim_{n\to\infty}\langle Af_n, g\rangle \\ &= \langle h, g\rangle.\end{aligned}$$

So, f is in the domain of $D(A^*) = D(A)$. Since g is arbitrary, we get $Af = A^*f = h = s\text{-}\lim_{n\to\infty} Af_n$. Therefore A is closed.