report

M1 Renta Yagi (Student ID:322201318)

1. (Example of a sequence on a Hilbert space which converges weakly but not strongly)
Let H be a infinite dimensional Hilbert space, $\{\varphi_n\}_{n=1}^{\infty}$be a complete orthonormal system of H. Then for any $x \in H$, we can get

$$\sum_{n=1}^{\infty} |\langle x, \varphi_n \rangle|^2 = \|x\|^2 < \infty$$

by Parseval's identity. Since a series of positive terms converges, we have $\lim_{n\to\infty} \langle x, \varphi_n \rangle = 0$. So $\{\varphi_n\}_{n=1}^{\infty}$ converges to 0 weakly. However, for any $n \in \mathbb{N}$, we have $\|\varphi_n\| = 1$. Therefore $\{\varphi_n\}_{n=1}^{\infty}$doesn't converge to 0 strongly.
2. (Example of a sequence of operator on a Hilbert space which strongly converges but does not converge uniformly)
Let $X = L^2(0, \infty), \{A_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ and

$$(A_n u)(x) = u(x+n) \quad (\forall u \in X, \forall x \in (0, \infty)).$$

Then

$$\begin{aligned} \|A_n u\|^2 &= \int_0^{\infty} |(A_n u)(x)|^2 \, dx \\ &= \int_0^{\infty} |u(x+n)|^2 \, dx \\ &= \int_n^{\infty} |u(x)|^2 \, dx \xrightarrow{n\to 0} 0. \end{aligned}$$

So $\{A_n\}_{n=1}^{\infty}$ converges to 0 strongly.
On the other hand, for $u \in L^2(0, \infty)$ which satisfies $u(x) = 0$ (if $x \leq n$) we have

$$\begin{aligned} \|A_n u\|^2 &= \int_0^{\infty} |u(x+n)|^2 \, dx \\ &= \int_n^{\infty} |u(x)|^2 \, dx \\ &= \int_0^{\infty} |u(x)|^2 \, dx \\ &= \|u\|^2. \end{aligned}$$

Since $\|A_n\| = \sup_{\|u\| \leq 1} \|A_n u\| = 1$, the sequence $\{A_n\}_{n=1}^{\infty}$ doesn't converge to 0 uniformly.