An example of Bayesian inference

PENG Qi

August 21, 2022

In this example we will be discussing the property of the distribution of observation variable Y,with a probability density function π_Y, assuming that X and ε be absolutely continuous. Then the relation among Y, X and ε is given by definition 5.1 of [RC], that is, for $X : \Omega \to \mathbb{R}^N$ and $\varepsilon : \Omega \to \mathbb{R}^M$ over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, having $Y : \Omega \to \mathbb{R}^M$ defined by

$$Y = H(X) + \varepsilon, \tag{1}$$

where $H : \mathbb{R}^N \to \mathbb{R}^M$ called the observation operator and assumed to be continuous.

Then consider the pdf of Y: from equation (1), one has

$$\pi_Y(y) = \int_{\mathbb{R}^N} \pi_\varepsilon(y - H(x))\pi_X(x)dx.$$

This is given under the consideration of conditional probability implying the joint distribution. The detail is shown in Page 133-134 of [RC].

Then, here we come to the explanation of Example 5.4 for the case in which X and ε are multivariate normal random variables and $\bar{x}$ the mean value of X and also the mean of ε be zero. In this example, we assume the observation operator $H : \mathbb{R}^N \to \mathbb{R}^M$ be the linear map H defined by

$$H(x) = Hx,$$

where H is a $(M \times N)$-dimensional matrix. Then we know that the pdf of π_ε and π_X satisfy normal distribution, and can be shown as the proportional relation:

$$\begin{aligned}
\pi_\varepsilon(\epsilon) &\propto exp(-\frac{1}{2}\epsilon^T R^{-1}\epsilon) \\
&\propto exp(-\frac{1}{2}(y - Hx)^T R^{-1}(y - Hx)) \\
\pi_X(x) &\propto exp(-\frac{1}{2}(x - \bar{x})^T P^{-1}(x - \bar{x})).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\pi_Y(y) &\propto \int_{\mathbb{R}^N} exp(-\frac{1}{2}(y - Hx)^T R^{-1}(y - Hx)) \times exp(-\frac{1}{2}(x - \bar{x})^T P^{-1}(x - \bar{x}))dx \\
&= \int_{\mathbb{R}^N} exp(-\frac{1}{2}[(y - Hx)^T R^{-1}(y - Hx) + (x - \bar{x})^T P^{-1}(x - \bar{x})])dx
\end{aligned}$$

Then, we introduce the completing-the-square formula which is shown as

$$x^T Cx - 2d^T x = (x - C^{-1}d)^T C(x - C^{-1}d) - d^T C^{-1}d,$$

Setting firstly

$$I = -\frac{1}{2}[(y - Hx)^T R^{-1}(y - Hx) + (x - \bar{x})^T P^{-1}(x - \bar{x})]$$

Then we reformulate the component as

$$I = -\frac{1}{2}[(x - C^{-1}d)^T C(x - C^{-1}d) - d^T C^{-1}d + y^T R^{-1}y + \bar{x}^T P^{-1}\bar{x}],$$

where

$$C = P^{-1} + H^T R^{-1} H,$$
$$d = H^T R^{-1} y + P^{-1} \bar{x}.$$

Here, some of the component I cancelled out as $(x - C^{-1}d)^T C(x - C^{-1}d) + \bar{x}^T P^{-1} \bar{x}$ are constant due to the integral, as we do the integration over the variable dx to give away the x dependence, so that we finally obtained that

$$\pi_Y(y) \propto exp(-\frac{1}{2}(y^T R^{-1} y - d^T C^{-1} d)) \int_{\mathbb{R}^N} exp(-\frac{1}{2}((x - C^{-1}d)^T C(x - C^{-1}d) + \bar{x}^T P^{-1} \bar{x}))dx$$

Therefore,

$$\pi_Y(y) \propto exp(-\frac{1}{2}(y^T R^{-1} y - d^T C^{-1} d)).$$

Reference

- [RC]: Probabilistic Forecasting and Bayesian Data Assimilation by S. Reich, C. Cotter, Page 132-134