数理科学特論I Report

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<u>Problem</u> If A is a self-adjoint element of a C^*-algebra $\mathcal{C}$, show that $r(A) = \|A\|$. //

<u>Proof</u> I will show this claim for "normal" element.

Let be A normal element in $\mathcal{C}$, i.e. $A^*A = AA^*$.

$$\forall n \in \mathbb{N},\ \|A^{2^n}\|^2 \underset{\boxed{C^*\text{-identity}}}{=} \|(A^{2^n})^*(A^{2^n})\| \underset{\boxed{\text{property of adjoint}}}{=} \|(A^*)^{2^n} A^{2^n}\|$$

$$\underset{\boxed{A^*A = AA^*}}{=} \|(A^*A)^{2^n}\| = \|(A^*A)^{2^{n-1}}\|^2 = \cdots = \|A^*A\|^{2^n} = \|A\|^{2\cdot 2^n}$$

C^*-identity

$$\therefore \forall n \in \mathbb{N},\ \|A^{2^n}\|^{\frac{1}{2^n}} = \|A\|.$$

By <u>Gelfand-Beurling formula</u>,

$$r(A) = \exists\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|A^{2^n}\|^{\frac{1}{2^n}} = \|A\|.$$

So, if A is a normal element of a C^*-algebra $\mathcal{C}$,

then $r(A) = \|A\|$.

In particular, self-adjoint elements is normal. //