Nagoya University, G30 program Spring 2022

Mathematics Tutorial IIa ♣ ♦ ♠ ♥ Instructor : Serge Richard

Homework 6

Exercise 1 (Spherical coordinates) *Consider the map* $\Phi : [0, \infty) \times [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$ *with*

$$\Phi(r, \theta, \varphi) := \big(r\cos(\theta)\sin(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\varphi)\big).$$

Compute the Jacobian matrix corresponding to this function.

Exercise 2 *Let us consider* $f : \mathbb{R}^3 \to \mathbb{R}$ *defined by* $f(x, y, z) = \mathrm{e}^{xy}\cos(z)$ *for any* $(x, y, z) \in \mathbb{R}^3$*. Assume also that* $x = tu$*,* $y = \sin(tu)$ *and* $z = u^2$ *for some* $t, u \in \mathbb{R}$*. By setting*

$$F(t, u) := f\big(tu, \sin(tu), u^2\big)$$

Compute the derivative $\partial_2 F \equiv \partial_u F$ *by three different methods: once by a direct computation, once as one component of the derivative of the composition of two functions (chain rule), and once with the formula often seen in the literature*

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.$$

Can you explain where this formula comes from ? Can you also understand the (horrible) formula

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}.$$

In the following exercises we call *a vector field* a function f defined on an open set $\Omega \subset \mathbb{R}^n$ and taking values in $\mathbb{R}^n$. In other words, for $\Omega \subset \mathbb{R}^n$ open, a function $f : \Omega \to \mathbb{R}^d$ is a vector field if $d = n$.

Exercise 3 *Provide a picture for the following vector fields:*

(i) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = \binom{1}{1}$,

(ii) $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = x\,E_1 + y\,E_2$,

(iii) $\nabla k : \mathbb{R}^2 \to \mathbb{R}^2$ *for* $k : \mathbb{R}^2 \to \mathbb{R}$ *given by* $k(x, y) = \frac{1}{1+x^2+y^2}$.

Let $f : \Omega \to \mathbb{R}^n$ be a vector field. If there exists a differentiable function $\phi : \Omega \to \mathbb{R}$ such that $f = \nabla\phi$ we say that ϕ is a *potential function for* f.

Exercise 4 *For the following functions, does a potential function exist ?*

(i) $f(x, y) = (y, x)$, *(ii)* $f(x, y) = (3x^2y + 2x + y^3, x^3 + 3xy^2 - 2y)$,

(iii) $f(x, y) = \big(\cos(x), \sin(y)\big)$, *(iv)* $f(x, y, z) = \big(x^2 - yz, y^2 - zx, z^2 - xy\big)$.