
Homework 5

Exercise 1 *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x,y) = x\,y\,\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}$.

(i) Study the local extrema of f *(you can use the symmetries of this function),*

(ii) Show that $f(x,y) \to 0$ *as* $\|(x,y)\| \to \infty$,

(iii) Deduce that there exist some global extrema and compute them.

So far we have considered the norm in $\mathbb{R}^n$ provided by the formula

$$\|X\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{j=1}^n x_j^2} =: \|X\|_2.$$

We could have considered other expressions, as for example

$$\|X\|_1 := |x_1| + |x_2| + \cdots + |x_n| = \sum_{j=1}^n |x_j|, \quad \text{or} \quad \|X\|_\infty := \max_{j\in\{1,\dots,n} |x_j|.$$

Exercise 2 *a) Show that the two alternative expressions also define norms. Namely, they satisfy the three conditions of any norm* $|||\cdot|||$ *:*

(i) $|||X||| \geq 0$ *for any* $X \in \mathbb{R}^n$, *and* $|||X||| = 0$ *if and only if* $X = \mathbf{0}$,

(ii) $|||\lambda X||| = |\lambda|\,|||X|||$, *for any* $\lambda \in \mathbb{R}$ *and* $X \in \mathbb{R}^n$,

(iii) $|||X + Y||| \leq |||X||| + |||Y|||$, *for any* $X, Y \in \mathbb{R}^n$.

b) Draw the unit ball of $\mathbb{R}^2$ *with the three norms* $\|\cdot\|_2$, $\|\cdot\|_1$ *and* $\|\cdot\|_\infty$.

Exercise 3 *Show that for any* $X \in \mathbb{R}^n$ *one has*

$$\|X\|_2 \leq \sqrt{n}\|X\|_\infty \leq \sqrt{n}\|X\|_1 \leq n\|X\|_2.$$

Exercise 4 *Let us consider* $f : \mathbb{R}^3 \to \mathbb{R}$ *and* $F : \mathbb{R}^3 \to \mathbb{R}^3$ *sufficiently differentiable on* $\mathbb{R}^3$. *One also sets* $F = {}^t(F_1, F_2, F_3)$ *and*

$$\begin{aligned}
\mathrm{grad}(f) &\equiv \nabla f = {}^t\big(\partial_1 f, \partial_2 f, \partial_3 f\big) \\
\mathrm{div}(F) &\equiv \nabla \cdot F = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 \\
\mathrm{curl}(F) &\equiv \nabla \times F = {}^t\big(\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1\big) \\
\Delta f &= \partial_1^2 f + \partial_2^2 f + \partial_3^2 f.
\end{aligned}$$

Show then the following relations:

(i) $\mathrm{div}(fF) = f\mathrm{div}(F) + \mathrm{grad}(f) \cdot F$,

(ii) $\mathrm{div}\big(\mathrm{curl}(F)\big) = 0$,

(iii) $\mathrm{curl}\big(\mathrm{grad}(f)\big) = 0$,

(iv) $\mathrm{curl}\big(\mathrm{curl}(F)\big) = \mathrm{grad}\big(\mathrm{div}(F)\big) - \Delta F$, *with* $\Delta F = {}^t(\Delta F_1, \Delta F_2, \Delta F_3)$.