
Homework 4

Exercise 1 (Geometrical interpretation of the gradient) *a) Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *given by* $f(x, y) = x^2 + 4y^2$.

(i) *Compute the gradient of* f *at any point* (x, y),

(ii) *For* $k > 0$, *describe the* k*-level set* L_k, *and for this level set express* y *as a function of* x *and* k,

(iii) *For any* $(x, y) \in \mathbb{R}^2$ *such that* $f(x, y) = k$, *show that the gradient of* f *at* (x, y) *is orthogonal to the curve described by* L_k.

b) More generally, let $f : \mathbb{R}^n \to \mathbb{R}$ *be a function of class* C^1 *and let* $X \in \mathbb{R}^n$ *such that* $[\nabla f](X) \neq \mathbf{0}$. *Let* $k \in \mathbb{R}$ *be given by* $k := f(X)$ *and consider the* k*-level set* L_k. *This* k*-level set can be considered (at least locally) as a surface of dimension* $n - 1$ *in* $\mathbb{R}^n$. *Show that* $[\nabla f](X)$ *is perpendicular to the surface* L_k. *For that purpose, we can consider any parametric curve* $\varphi : (-1, 1) \to L_k$ *with* $\varphi(0) = X$ *and show that* $[\nabla f](X)$ *is perpendicular to it at the point* X.

Exercise 2 (i) *Compute the Taylor expansion around* $(0, 0)$ *and up to the second order of the function*

$$\mathbb{R}^2 \ni (x, y) \mapsto \mathrm{e}^{x^2+xy+y^2} \in \mathbb{R}.$$

(ii) *Compute the Taylor expansion around* $(0, 0)$ *up to the third order of the function*

$$\mathbb{R}^2 \ni (x, y) \mapsto \mathrm{e}^{x+y} \in \mathbb{R}.$$

By fixing then $x = y = 1/2$ *in the polynomial you have obtained, what can you say about the number* e *?*

Exercise 3 *Consider the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *defined by* $f(x, y) = 2x^3 + 6xy - 3y^2 + 2$ *for any* $(x, y) \in \mathbb{R}^2$.

(i) *Determine the local extrema of* f,

(ii) *Does* f *possess global extrema ?*

(iii) *Consider the segment* L *defined by*

$$L = \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 0,\ y = x + 1\}$$

and determine the global extrema of f *restricted to* L. *Where are these extrema located ?*