Homework 2

Exercise 1 *The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a* cycloid. *Assume that the circle has radius r and that the point P is initially located at the origin of the x-axis.*

(i) Determine the parametric curve defined by the point P,

(ii) Determine the tangent line at any point of the cycloid,

(iii) When is this tangent line horizontal or vertical ?

(iv) Find the area under one arch of the cycloid,

(v) Find the length of one arch of the cycloid.

Exercise 2 *Let $\Omega \subset \mathbb{R}^2$ and consider the functions $f_i : \Omega \to \mathbb{R}$ defined for $(x, y) \in \Omega$ by*

$$a)\ f_1(x,y) = xy, \quad b)\ f_2(x,y) = (x+1)(y+3) \quad c)\ f_3(x,y) = \frac{xy}{x^2+y^2} \quad d)\ f_4(x,y) = \frac{x+y}{x-y}.$$

1. *Determine the maximal domain Ω on which these functions are well defined,*
2. *Sketch the k-level sets for these functions.*

Exercise 3 *Consider the following functions defined on $\mathbb{R}^2 \setminus \{(0,0)\}$*

$$a)\ f_1(x,y) = \frac{x^2-y^2}{x^2+y^2}, \quad b)\ f_2(x,y) = \frac{xy}{x^2+y^2}, \quad c)\ f_3(x,y) = \frac{1}{x^2+y^2+1}.$$

For each of them compute the limits $\lim_{x\to 0}\big(\lim_{y\to 0} f_i(x,y)\big)$, $\lim_{y\to 0}\big(\lim_{x\to 0} f_i(x,y)\big)$, and $\lim_{(x,y)\to(0,0)} f_i(x,y)$. Discuss your result.

Exercise 4 *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x,y) := \begin{cases} \frac{x^2 y}{x^4 - 2x^2 y + 3y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

1. *Study the limit $(x,y) \to (0,0)$ along the path of equation $y = mx$ for any $m \in \mathbb{R}$,*
2. *Study the limit $(x,y) \to (0,0)$ along the path of equation $y = x^2$,*
3. *Show that f is not continuous at $(0,0)$.*