
Homework 13

Exercise 1 Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be sufficiently many times differentiable and satisfying the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

(i) Let c be a closed parametric curve oriented counterclockwise and non-intersecting. Show that

$$\int_c \begin{pmatrix} \partial_y f \\ -\partial_x f \end{pmatrix} = 0,$$

(ii) Show that $f(0,0) = \frac{1}{2\pi} \int_0^{2\pi} f(r \cos(\theta), r \sin(\theta)) d\theta$ for any $r > 0$.

Exercise 2 (Final exam 2020) The aim of this exercise is to show that the area of a surface is independent of its parametrization. Consider an open set $\Omega \subset \mathbb{R}^2$, a function $g : \Omega \rightarrow \mathbb{R}^3$ of class C^1 , and the parametric surface given by

$$\Omega \ni (s, t) \mapsto q(s, t) = \begin{pmatrix} s \\ t \\ g(s, t) \end{pmatrix} \in \mathbb{R}^3$$

Let also $\varphi : \Lambda \ni (x, y) \mapsto (\varphi_1(x, y), \varphi_2(x, y)) \in \Omega$ be a diffeomorphism of class C^1 .

- (i) Write the formula for the computation of the area of $q(\Omega)$,
- (ii) Compute $[\partial_1(g \circ \varphi)](x, y)$ and $[\partial_2(g \circ \varphi)](x, y)$ (altogether you should obtain 4 terms),
- (iii) Using the previous information, compute the vectors $[\partial_1(q \circ \varphi)](x, y)$ and $[\partial_2(q \circ \varphi)](x, y)$,
- (iv) Compute $\left([\partial_1(q \circ \varphi)](x, y)\right) \times \left([\partial_2(q \circ \varphi)](x, y)\right)$, and factor the common term $\left[(\partial_1 \varphi_1)(\partial_2 \varphi_2) - (\partial_1 \varphi_2)(\partial_2 \varphi_1)\right](x, y)$,
- (v) Conclude with the general formula for the change of variables for a volume integral that the area of a surface is independent of its parametrization, or more precisely show that the area of $q(\Omega)$ is equal to the area of $[q \circ \varphi](\Lambda)$.

Exercise 3 (Final exam 2018) Consider the set $\Omega := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$ and the surface S defined by

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid z = 9 - x^2 - y^2 \text{ for } (x, y) \in \Omega\}.$$

1. Sketch the surface S ,
2. Provide a parametrization of the surface S ,
3. If $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the vector field given by $\Psi(x, y, z) = (2z - y, x + z, 3x - 2y)$, compute the flux of $\text{curl}(\Psi)$ through the surface S , with the normal vector on S pointing outside of the surface,
4. Compute the integral of Ψ along the boundary of S and check Stokes's theorem on this example.