Homework 12

Exercise 1 *Consider the parametrization of the sphere of radius $r > 0$ given by $q : [0, 2\pi) \times [0, \pi) \to \mathbb{R}^3$ with*

$$q(\theta, \varphi) := \begin{pmatrix} r\cos(\theta)\sin(\varphi) \\ r\sin(\theta)\sin(\varphi) \\ r\cos(\varphi) \end{pmatrix}.$$

Compute the vectors $[\partial_1 q](\theta, \varphi)$, $[\partial_2 q](\theta, \varphi)$, and the vector normal to the sphere at the point $q(\theta, \varphi)$.

Exercise 2 *Let $g : [0, 1] \to \mathbb{R}_+$ of class C^1 and consider the surface of revolution defined by*

$$q : [0, 1] \times [0, 2\pi) \ni (x, \theta) \mapsto \begin{pmatrix} x \\ g(x)\cos(\theta) \\ g(x)\sin(\theta) \end{pmatrix} \in \mathbb{R}^3.$$

Compute the area of this surface.

Exercise 3 *Let $\Omega \subset \mathbb{R}^2$ be open and let $g : \Omega \to \mathbb{R}$ be of class C^1. We consider the surface of $\mathbb{R}^3$ parameterized by the function $q : \Omega \to \mathbb{R}^3$ defined by $q(x, y) = {}^t\big(x, y, g(x, y)\big)$. Compute the area of the surface $q(\Omega)$. Consider also the surfaces defined by*

(i) *Ω is the disc of radius 1 centered at $(0, 0) \in \mathbb{R}^2$ and $g(x, y) = x^2 + y^2$,*

(ii) *Ω is the disc of radius 1 centered at $(0, 0) \in \mathbb{R}^2$ and $g(x, y) = xy$,*

Exercise 4 *Consider the upper half-sphere S in $\mathbb{R}^3$ centered in $(0, 0, 0)$ and of radius R, and let $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x, y, z) = x^2 + y^2$. Compute the integral $\iint_S f \,\mathrm{d}\sigma$ of f on the upper half-sphere. Same question for f defined by $f(x, y, z) = (x^2 + y^2)z$.*

Exercise 5 *Consider the vector field f in $\mathbb{R}^3$ defined by $f(x, y, z) = (x, y, 0)$. Compute the flux of this vector field through the sphere in $\mathbb{R}^3$ centered at $(0, 0, 0)$ and of radius r.*