

Homework 11

Exercise 1 Compute the area of the domain defined in $\mathbb{R}_+ \times \mathbb{R}_+$ by the four curves of equation

$$y = ax, \quad y = x/a, \quad y = b/x, \quad y = 1/bx \quad \text{for } a > 1, b > 1.$$

Let Ω be a body in \mathbb{R}^n and let $\varrho : \Omega \rightarrow \mathbb{R}_+$ denote its density function (for $X \in \Omega$ the value $\varrho(X)$ denotes the density of Ω at X). Let M denote the total mass of the body, and let \bar{X} denote the coordinates of its center of mass. These quantities are defined by

$$M = \int_{\Omega} \varrho(X) dX,$$
$$\bar{X} = \frac{1}{M} \int_{\Omega} X \varrho(X) dX.$$

Note that the second line represents in fact n equalities.

Exercise 2 (i) Find the center of mass of the quarter of a unit disc Ω defined in polar coordinates by

$$\Omega = \{(r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 \mid 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq \pi/2\},$$

(ii) Find the z -coordinate of the center of mass of the upper half of a unit ball centered at $0 \in \mathbb{R}^3$.

Exercise 3 Use Green's theorem to compute the integral $\int_c f$ with $f(x, y) = (y^2, x)$ when c corresponds to the following curves, taken counterclockwise:

(i) The square of vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$,

(ii) The square of vertices $(\pm 1, \pm 1)$,

(iii) The circle of radius 1 and centered at $(0, 0)$,

(iv) The ellipse of equation $(x/a)^2 + (y/b)^2 = 1$ for some $a, b > 0$.

Exercise 4 Check the validity of Green's theorem for the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined for $(x, y) \in \mathbb{R}^2$ by $f(x, y) = (2xy, x^2)$ on the domain $\Omega = [-1, 2] \times [-1, 3] \subset \mathbb{R}^2$.