Graphs and epidemiology

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**Exercise**

Study some examples of lumping as presented in Sections 2.4, 2.5 of [KMS].

Example 1 ([KMS] Exercise 2.21)

Consider the simple network of two nodes connected by an edge.

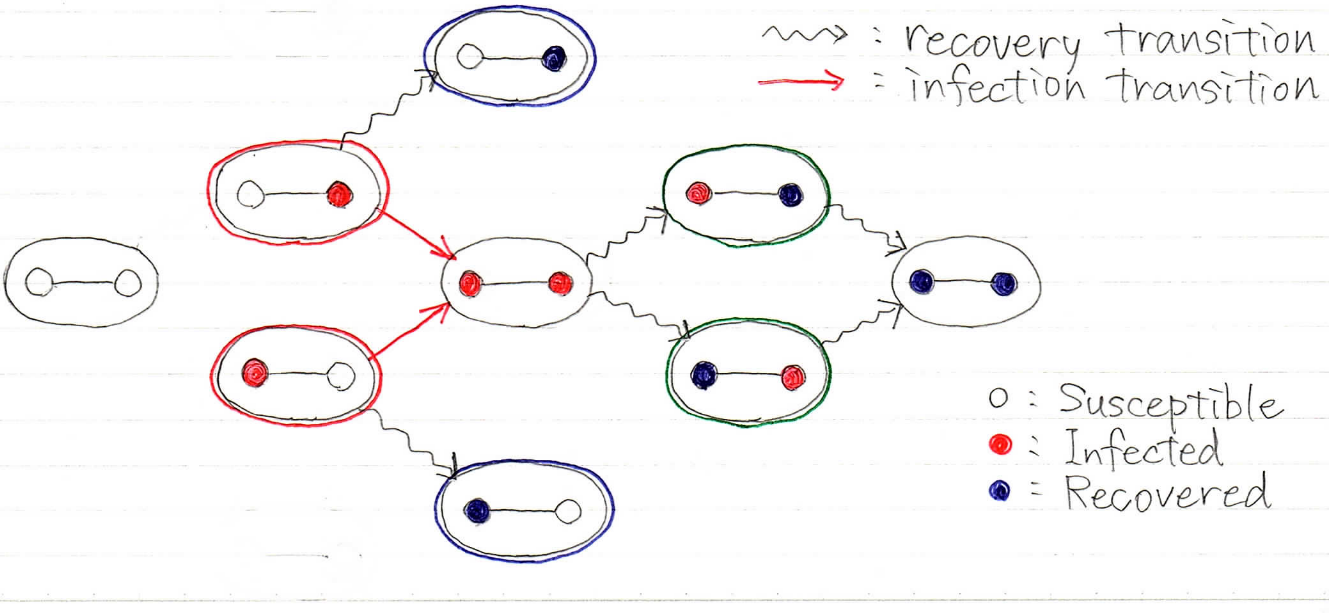
① — ②

We investigate SIR on this model.

The state space is partitioned into $C^{00}, C^{10}, C^{01}, C^{11}, C^{20}$, and $C^{02}$, where

$$C^{00}=\{SS\},\quad \underline{C^{10}=\{SI, IS\}},\quad \underline{C^{01}=\{SR, RS\}},$$

$$\underline{C^{11}=\{IR, RI\}},\quad C^{20}=\{II\},\quad C^{02}=\{RR\}$$



Then, the prob. $Y^{ik}$ of the class $C^{ik}$ is given by

$$Y^{ik} = \sum_{s \in C^{ik}} X_s ,$$

so it follows that

$$\dot{Y}_{00} = \dot{X}_{SS} = 0 ,$$

$$\dot{Y}_{10} = -(\gamma+\tau) Y_{10} ,$$

$$\dot{Y}_{01} = \gamma Y_{10} ,$$

$$\dot{Y}_{20} = \tau Y_{10} - 2\gamma Y_{20} ,$$

$$\dot{Y}_{11} = 2\gamma Y_{20} - \gamma Y_{11} ,$$

$$\dot{Y}_{02} = \gamma Y_{11} .$$

($\gamma$ : recovery rate, $\tau$ : transmission rate)

Let

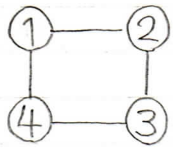
$$Y := {}^t( Y_{00}, Y_{10}, Y_{01}, Y_{20}, Y_{11}, Y_{02})$$

then $\dot{Y} = AY$, where

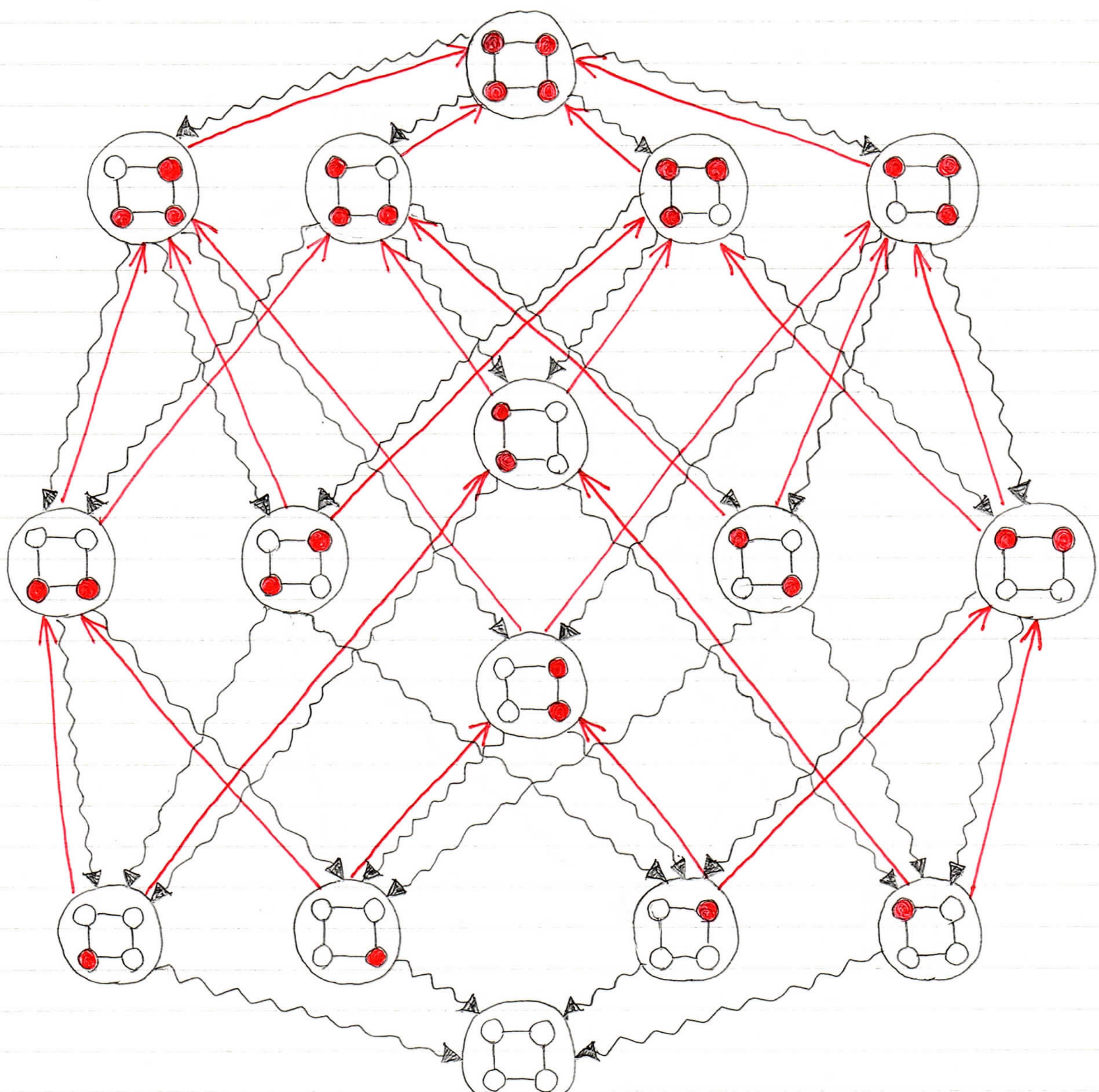
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\gamma+\tau) & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & -2\gamma & 0 & 0 \\ 0 & 0 & 0 & 2\gamma & -\gamma & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \end{pmatrix} .$$

Example 2 ([KMS] Exercise 2.17)

Consider a cycle graph with $N=4$ nodes.




We investigate SIS on this graph. The state transition diagram is as follows.



Paying attention to the symmetry of the graph, we consider the following partition of the state space.

$\mathcal{C}^0 = \{SSSS\}$, $\mathcal{C}^1 = \{SSSI, SSIS, SISS, ISSS\}$,

$\mathcal{C}^{2A} = \{IISS, SIIS, SSII, ISSI\}$,

$\mathcal{C}^{2B} = \{SISI, ISIS\}$

$\mathcal{C}^3 = \{SIII, ISII, IISI, IIIS\}$, $\mathcal{C}^4 = \{IIII\}$

Then, the prob. $Y^Z$ of the class $\mathcal{C}^Z$ is given by

$$Y^Z = \sum_{S \in \mathcal{C}^Z} X_S,$$

so it follows that

$$\dot{Y}_0 = \gamma Y_1,$$

$$\dot{Y}_1 = 2\gamma(Y_{2A} + Y_{2B}) - (\gamma + 2\tau) Y_1$$
$$\left( \begin{array}{l} \because \dot{X}_{SSSI} = \gamma(X_{SSII} + X_{ISSI}) + \gamma X_{SISI} - (\gamma + 2\tau) X_{SSSI} \\ +4 \text{ similar equations} \end{array} \right),$$

$$\dot{Y}_{2A} = 2\gamma Y_3 + 2\tau Y_1 - (2\gamma + 2\tau) Y_{2A}$$
$$\left( \begin{array}{l} \because \dot{X}_{SSII} = \gamma(X_{SIII} + X_{ISII}) + \tau(X_{SSSI} + X_{SSIS}) \\ \qquad - (2\tau + 2\gamma) X_{SSII} \\ +4 \text{ similar equations} \end{array} \right),$$

$$\dot{Y}_{2B} = \gamma Y_3 - (2\gamma + 4\tau) Y_{2B}$$
$$\left( \begin{array}{l} \because \dot{X}_{SISI} = \tau(X_{SIII} + X_{IISI}) - (2\gamma + 4\tau) X_{SISI} \\ \dot{X}_{ISIS} = \tau(X_{IIIS} + X_{ISII}) - (2\gamma + 4\tau) X_{ISIS} \end{array} \right),$$

$$\dot{Y}_3 = 4\gamma Y_4 + 2\tau Y_{2A} + 4\tau Y_{2B} - (3\gamma + 2\tau) Y_3$$
$$\left( \begin{array}{l} \because \dot{X}_{SIII} = \gamma Y_4 + \tau(X_{SSII} + X_{SIIS}) + 2\tau X_{SISI} \\ \qquad - (3\gamma + 2\tau) X_{SIII} \\ +4 \text{ similar equations} \end{array} \right),$$

$$\dot{Y}_4 = 2\tau Y_3 - 4\gamma Y_4.$$

Let $Y := {}^t(Y_0, Y_1, Y_{2A}, Y_{2B}, Y_3, Y_4)$.

Then $\dot{Y} = AY$, where

$$A = \begin{pmatrix} 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & -(\gamma+2\tau) & 2\gamma & 2\gamma & 0 & 0 \\ 0 & 2\tau & -(2\gamma+2\tau) & 0 & 2\gamma & 0 \\ 0 & 0 & 0 & -(2\gamma+4\tau) & \gamma & 0 \\ 0 & 0 & 2\tau & 4\tau & -(3\gamma+2\tau) & 4\gamma \\ 0 & 0 & 0 & 0 & 2\tau & -4\gamma \end{pmatrix}.$$