Graphs and epidemiology

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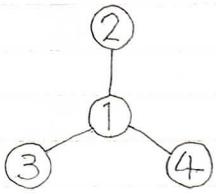
Exercise (P. 26)

Study some examples of lumping as presented in Sections 2.4, 2.5 of [KMS].

∘ Example ([KMS] Example 2.8 & Exercise 2.10)

Consider the graph below.



We investigate SIS on this graph.

Let S_{ABCD} the state in which the node 1 has status A, the node 2 status B, the node 3 status C, the node 4 status D. $(A, B, C, D \in \{S, I\})$

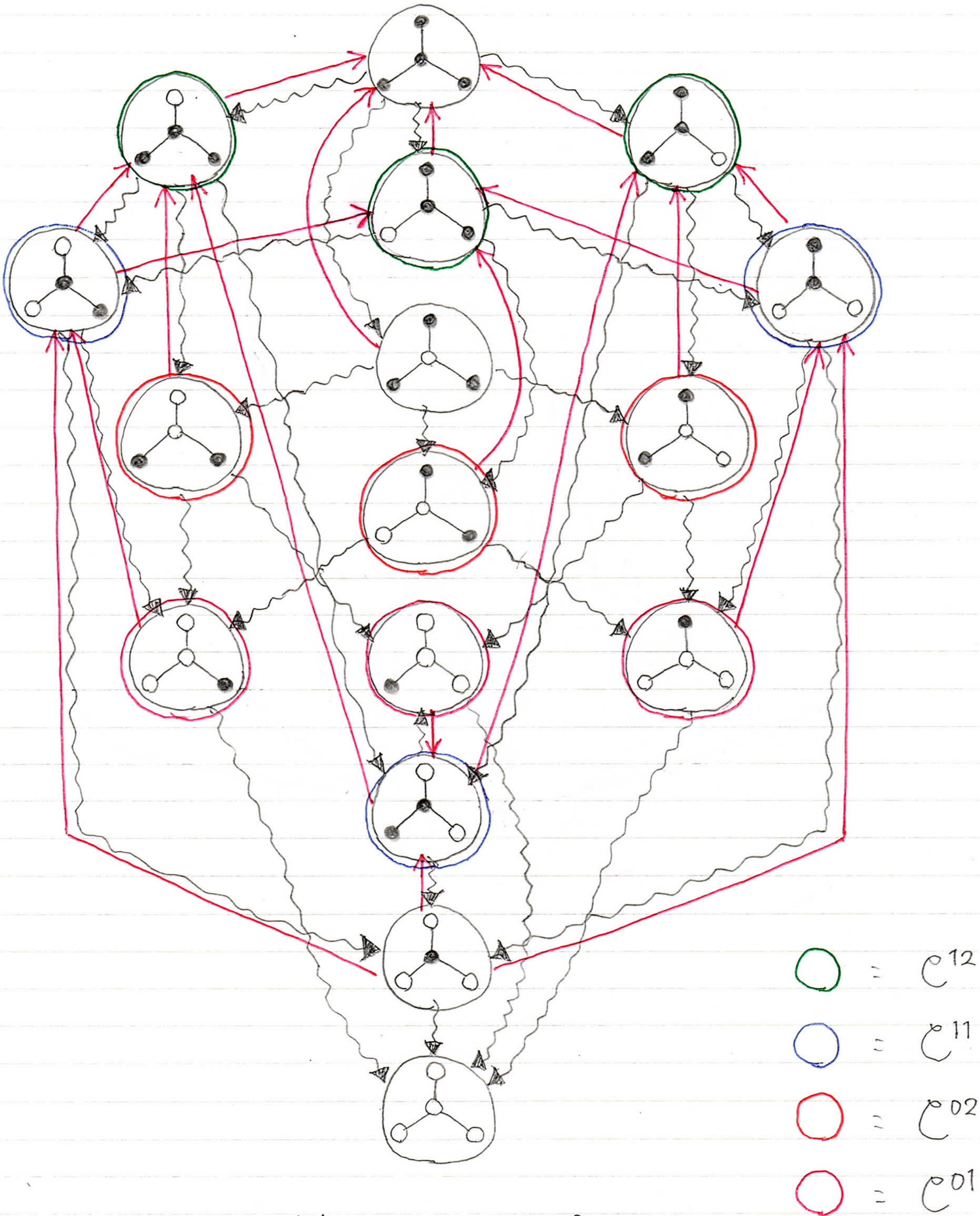
Then, we classify the states of the graph by focusing on the status of 1 and the number of infected nodes.

$C^{00} = \{S_{SSSS}\}$, $C^{10} = \{S_{ISSS}\}$

$C^{01} = \{S_{SISS}, S_{SSIS}, S_{SSSI}\}$, $C^{02} = \{S_{SIIS}, S_{SSII}, S_{SISI}\}$

$C^{11} = \{S_{IISS}, S_{ISIS}, S_{ISSI}\}$, $C^{12} = \{S_{IIIS}, S_{ISII}, S_{IISI}\}$

$C^{03} = \{S_{SIII}\}$, $C^{13} = \{S_{IIII}\}$



o : Susceptible, • : Infected,

⇝ : recovery transition,

⟶ : infection transition.

Then, the prob. Y^{ik} of the class C^{ik} is given as below.

$$Y^{00} = X_{SSSS}, \quad Y^{10} = X_{ISSS},$$

$$Y^{01} = X_{SISS} + X_{SSIS} + X_{SSSI}, \quad Y^{02} = X_{SIIS} + X_{SSII} + X_{SISI},$$

$$Y^{11} = X_{IISS} + X_{ISIS} + X_{ISSI}, \quad Y^{12} = X_{IIIS} + X_{ISII} + X_{IISI},$$

$$Y^{03} = X_{SIII}, \quad Y^{13} = X_{IIII}.$$

Now, it follows that

$$\dot{Y}^{00} = \gamma(Y^{10} + Y^{01}),$$

$$\dot{Y}^{10} = \gamma Y^{11} - (\gamma + 3\tau) Y^{10},$$

$$\dot{Y}^{03} = \gamma Y^{13} - (3\gamma + 3\tau) Y^{03},$$

$$\dot{Y}^{13} = 3\tau Y^{03} + \tau Y^{12} - 4\gamma Y^{13},$$

$$\dot{Y}^{01} = 2\gamma Y^{02} + \gamma Y^{11} - (\gamma + \tau) Y^{01},$$

$$\left(\because \dot{X}_{SISS} = \gamma(X_{SIIS} + X_{SISI}) + X_{IISS} - (\gamma + \tau) X_{SISS} \atop + 2 \text{ similar equations} \right)$$

$$\dot{Y}^{02} = 3\gamma Y^{03} + \gamma Y^{12} - (2\gamma + 2\tau) Y^{02},$$

$$\left(\because \dot{X}_{SIIS} = \gamma X_{SIII} + \gamma X_{IIIS} - (2\gamma + 2\tau) X_{SIIS} \atop + 2 \text{ similar equations} \right)$$

$$\dot{Y}^{11} = 3\tau Y^{10} + \tau Y^{01} + 2\gamma Y^{12} - (2\gamma + 2\tau) Y^{11},$$

$$\left(\begin{array}{l} \because \dot{X}_{IISS} = \tau X_{ISSS} + \tau X_{SISS} + \gamma(X_{IIIS} + X_{IISI}) \\ \qquad\qquad - (2\gamma + 2\tau) X_{IISS} \\ + 2 \text{ similar equations} \end{array} \right)$$

$$\dot{Y}^{12} = 3\gamma Y^{13} + 2\tau(Y^{02} + Y^{11}) - (3\gamma + \tau) Y^{12}.$$

$$\left(\begin{array}{l} \because\ \dot{X}_{IIIS} = \gamma X_{IIII} + 2\tau X_{SIIS} + \tau(X_{IISS} + X_{ISIS}) \\ \qquad\qquad\qquad\qquad\qquad\qquad - (3\gamma + \tau) X_{IIIS} \\ + \text{ 2 similar equations} \end{array} \right)$$

(γ: recovery rate, τ: transmission rate).

By the way, consider a different partition of the states:

$C^0 = \{S_{SSSS}\}$,

$C^1 = \{S_{ISSS},\ S_{SISS},\ S_{SSIS},\ S_{SSSI}\}$,

$C^2 = \{S_{IISS},\ S_{ISIS},\ S_{ISSI},\ S_{SIIS},\ S_{SISI},\ S_{SSII}\}$,

$C^3 = \{S_{IIIS},\ S_{IISI},\ S_{ISII},\ S_{SIII}\}$,

$C^4 = \{S_{IIII}\}$.

And, let Y_k be the prob. of the class C^k.

Then,

$$\begin{aligned} \dot{Y}_2 &= \dot{Y}^{02} + \dot{Y}^{11} \\ &= 3\gamma(Y^{03} + Y^{12}) + 3\tau Y^{10} + \tau Y^{01} - (2\gamma + 2\tau)(Y^{02} + Y^{11}) \\ &= 3\gamma Y_3 - (2\gamma + 2\tau) Y_2 + 3\tau Y^{10} + \tau Y^{01} \end{aligned}$$

so we cannot write down a system of linear differential equations just in terms of Y_k.

Let's go back to the arguement about the partition $\bigsqcup C^{ik}$.

Let

$$Y := {}^t(Y^{00}, Y^{10}, Y^{03}, Y^{13}, Y^{01}, Y^{02}, Y^{11}, Y^{12})$$

and

$$T := \begin{pmatrix} 0 & \gamma & 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & -(\gamma+3\tau) & 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & -3(\gamma+\tau) & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\tau & -4\gamma & 0 & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 & -(\gamma+\tau) & 2\gamma & \gamma & 0 \\ 0 & 0 & 3\gamma & 0 & 0 & -2(\gamma+\tau) & 0 & \gamma \\ 0 & 3\tau & 0 & 0 & \tau & 0 & -2(\gamma+\tau) & 2\gamma \\ 0 & 0 & 0 & 3\gamma & 0 & 2\tau & 2\tau & -(3\gamma+\tau) \end{pmatrix}$$

then $\dot{Y} = TY$.

Reference :

I. Kiss, J. Miller, P. Simon,

Mathematics of epidemics on networks,

Springer, 2017.