Graphs and epidemiology

Report 2.

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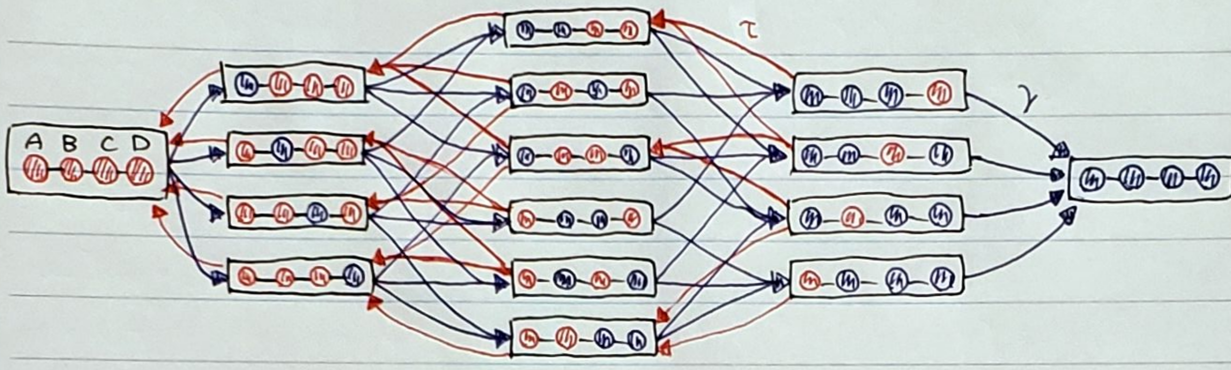
Exercise 2.5 ([KMS] P.38)



Ⓢ : susceptible, Ⓘ : infected, τ : transmission rate, γ : recovery rate.

· Set X_{ABCD} := "prob. that the system is in state ABCD."

One can get the following master equations from the figure;

$$
(*)\left\{
\begin{aligned}
\dot{X}_{SSSS} &= \gamma(X_{SSSI} + X_{SSIS} + X_{SISS} + X_{ISSS}) \\
\dot{X}_{SSSI} &= \gamma(X_{SSII} + X_{SISI} + X_{ISSI}) - (\tau+\gamma)X_{SSSI} \\
\dot{X}_{SSIS} &= \gamma(X_{SSII} + X_{SIIS} + X_{ISIS}) - (2\tau+\gamma)X_{SSIS} \\
\dot{X}_{SISS} &= \gamma(X_{SISI} + X_{SIIS} + X_{IISS}) - (2\tau+\gamma)X_{SISS} \\
\dot{X}_{ISSS} &= \gamma(X_{ISSI} + X_{ISIS} + X_{IISS}) - (\tau+\gamma)X_{ISSS} \\
\dot{X}_{SSII} &= \gamma(X_{SIII} + X_{ISII}) + \tau(X_{SSSI} + X_{SSIS}) - (\tau+2\gamma)X_{SSII} \\
\dot{X}_{SISI} &= \gamma(X_{SIII} + X_{IISI}) - (3\tau+2\gamma)X_{SISI} \\
\dot{X}_{ISSI} &= \gamma(X_{ISII} + X_{IISI}) - (2\tau+2\gamma)X_{ISSI} \\
\dot{X}_{SIIS} &= \gamma(X_{SIII} + X_{IIIS}) + \tau(X_{SISS} + X_{SSIS}) - (2\tau+2\gamma)X_{SIIS} \\
\dot{X}_{ISIS} &= \gamma(X_{ISII} + X_{IIIS}) - (3\tau+2\gamma)X_{ISIS} \\
\dot{X}_{IISS} &= \gamma(X_{IIIS} + X_{IISI}) + \tau(X_{SISS} + X_{ISSS}) - (\tau+2\gamma)X_{IISS} \\
\dot{X}_{SIII} &= \gamma X_{IIII} + \tau(X_{SSII} + X_{SIIS}) + 2\tau X_{SISI} - (\tau+3\gamma)X_{SIII} \\
\dot{X}_{ISII} &= \gamma X_{IIII} + \tau(X_{ISSI} + X_{ISIS}) - (2\tau+3\gamma)X_{ISII}
\end{aligned}
\right.
$$

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$$\left|\begin{array}{l}
\dot{X}_{IISI} = \gamma X_{IIII} + \tau (X_{SISI} + X_{ISSI}) - (2\tau + 3\gamma) X_{IISI} \\
\dot{X}_{IIIS} = \gamma X_{III} + \tau (X_{SIIS} + X_{IISS}) + 2\tau X_{ISIS} - (\tau + 3\gamma) X_{IIIS} \\
\dot{X}_{IIII} = \tau (X_{SIII} + X_{IIIS}) + 2\tau (X_{ISII} + X_{IISI}) - 4\gamma X_{IIII}.
\end{array}\right.$$

Put $X := {}^{t}(X_{SSSS}, X_{SSSI}, X_{SSIS}, X_{SISS}, X_{ISSS}, X_{SSII}, X_{SISI}, X_{ISSI}, X_{SIIS}, X_{ISIS}, X_{IISS}, X_{SIII}, X_{ISII}, X_{IISI}, X_{IIIS}, X_{IIII})$,

the we can rewrite (*) to the following; $\dot{X} = PX$ where

$$P := \begin{pmatrix}
0 & \gamma & \gamma & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\tau-\gamma & 0 & 0 & 0 & \gamma & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2\tau-\gamma & 0 & 0 & \gamma & 0 & 0 & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2\tau-\gamma & 0 & 0 & \gamma & 0 & \gamma & 0 & \gamma & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\tau-\gamma & 0 & 0 & \gamma & 0 & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 \\
0 & \tau & \tau & 0 & 0 & -\tau-2\gamma & 0 & 0 & 0 & 0 & 0 & \gamma & \gamma & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3\tau-2\gamma & 0 & 0 & 0 & 0 & \gamma & 0 & \gamma & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 & 0 & 0 & \gamma & \gamma & 0 & 0 \\
0 & 0 & \tau & \tau & 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 & \gamma & 0 & 0 & \gamma & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3\tau-2\gamma & 0 & 0 & \gamma & 0 & \gamma & 0 \\
0 & 0 & 0 & \tau & \tau & 0 & 0 & 0 & 0 & 0 & -\tau-2\gamma & 0 & 0 & \gamma & \gamma & 0 \\
0 & 0 & 0 & 0 & 0 & \tau & 2\tau & 0 & \tau & 0 & 0 & -\tau-3\gamma & 0 & 0 & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau & 0 & \tau & 0 & 0 & -2\tau-3\gamma & 0 & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & \tau & \tau & 0 & 0 & 0 & 0 & 0 & -2\tau-3\gamma & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau & 2\tau & \tau & 0 & 0 & 0 & -\tau-3\gamma & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau & 2\tau & 2\tau & \tau & -4\gamma
\end{pmatrix}.$$

Let $B^0 = (0)$, $C^0 = (\gamma, \gamma, \gamma, \gamma)$, $A^1 := (0, 0, 0, 0)$,

$$B^1 = \begin{pmatrix} -\tau-\gamma & 0 & 0 & 0 \\ 0 & -2\tau-\gamma & 0 & 0 \\ 0 & 0 & -2\tau-\gamma & 0 \\ 0 & 0 & 0 & -\tau-\gamma \end{pmatrix}, \quad C^1 = \begin{pmatrix} \gamma & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & 0 & 0 & \gamma & \gamma & 0 \\ 0 & \gamma & 0 & \gamma & 0 & \gamma \\ 0 & 0 & \gamma & 0 & \gamma & \gamma \end{pmatrix},$$

$$A^2 = \begin{pmatrix} \tau & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \tau & \tau & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \tau & \tau \end{pmatrix}, \quad B^2 = \begin{pmatrix} -\tau-2\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & -3\tau-2\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\tau-2\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & -3\tau-2\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & -\tau-2\gamma \end{pmatrix}, \quad C^2 = \begin{pmatrix} \gamma & \gamma & 0 & 0 \\ \gamma & 0 & \gamma & 0 \\ 0 & \gamma & \gamma & 0 \\ \gamma & 0 & 0 & \gamma \\ 0 & \gamma & 0 & \gamma \\ 0 & 0 & \gamma & \gamma \end{pmatrix},$$

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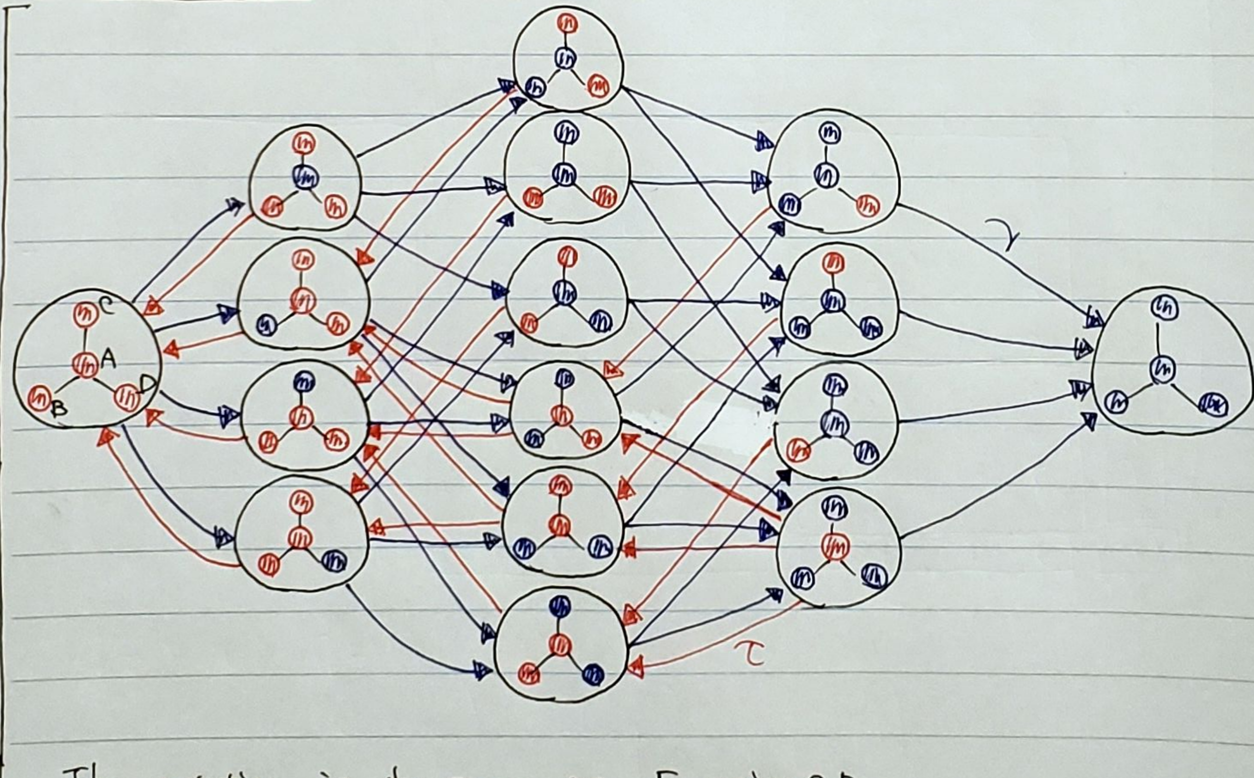
$$A^3 = \begin{pmatrix} \tau & 2\tau & 0 & \tau \\ 0 & 0 & \tau & 0 \\ 0 & \tau & \tau & 0 \\ 0 & 0 & 0 & \tau \end{pmatrix}, \quad B^3 = \begin{pmatrix} -\tau-3\gamma & 0 & 0 & 0 \\ 0 & -2\tau-3\gamma & 0 & 0 \\ 0 & 0 & -2\tau-3\gamma & 0 \\ 0 & 0 & 0 & -\tau-3\gamma \end{pmatrix}, \quad C^3 = \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix},$$

$A^4 = (\tau, 2\tau, 2\tau, \tau)$, $B^4 = (-4\gamma)$, then we obtain the block diagonal form of P;

$$P = \begin{pmatrix} B^0 & C^0 & & & \\ A^1 & B^1 & C^1 & & O \\ & A^2 & B^2 & C^2 & \\ O & & A^3 & B^3 & C^3 \\ & & & A^4 & B^4 \end{pmatrix}. \qquad \square$$

Exercise 2.6



The notation is the same as Exercise 2.5.

We have the master equations for this system from the figue;

$$\begin{cases} \dot{X}_{SSSS} = \gamma(X_{SSSI} + X_{SSIS} + X_{SISS} + X_{ISSS}) \end{cases}$$

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$$
\left\{
\begin{aligned}
\dot{X}_{SSSI} &= \gamma(X_{SSII} + X_{SISI} + X_{ISSI}) - (\tau+\gamma)X_{SSSI} \\
\dot{X}_{SSIS} &= \gamma(X_{SSII} + X_{SIIS} + X_{ISIS}) - (\tau+\gamma)X_{SSIS} \\
\dot{X}_{SISS} &= \gamma(X_{SISI} + X_{SIIS} + X_{IISS}) - (\tau+\gamma)X_{SISS} \\
\dot{X}_{ISSS} &= \gamma(X_{ISSI} + X_{ISIS} + X_{IISS}) - (3\tau+\gamma)X_{ISSS} \\
\dot{X}_{SSII} &= \gamma(X_{ISII} + X_{SIII}) - (2\tau+2\gamma)X_{SSII} \\
\dot{X}_{SISI} &= \gamma(X_{SIII} + X_{IISI}) - (2\tau+2\gamma)X_{SISI} \\
\dot{X}_{ISSI} &= \gamma(X_{IISI} + X_{ISII}) - (2\tau+2\gamma)X_{ISSI} + \tau(X_{SSSI} + X_{ISSS}) \\
\dot{X}_{SIIS} &= \gamma(X_{SIII} + X_{IIIS}) - (2\tau+2\gamma)X_{SIIS} \\
\dot{X}_{ISIS} &= \gamma(X_{ISII} + X_{IIIS}) + \tau(X_{SSIS} + X_{ISSS}) - (2\tau+2\gamma)X_{ISIS} \\
\dot{X}_{IISS} &= \gamma(X_{IISI} + X_{IIIS}) + \tau(X_{ISSS} + X_{SISS}) - (2\tau+2\gamma)X_{IISS} \\
\dot{X}_{SIII} &= \gamma X_{IIII} - (3\tau+3\gamma)X_{SIII} \\
\dot{X}_{ISII} &= \gamma X_{IIII} + \tau(X_{ISSI} + X_{ISIS}) + 2\tau X_{SSII} - (\tau+3\gamma)X_{ISII} \\
\dot{X}_{IISI} &= \gamma X_{IIII} + \tau(X_{ISSI} + X_{IISS}) + 2\tau X_{SISI} - (\tau+3\gamma)X_{IISI} \\
\dot{X}_{IIIS} &= \gamma X_{IIII} + \tau(X_{ISIS} + X_{IISS}) + 2\tau X_{SIIS} - (\tau+3\gamma)X_{IIIS} \\
\dot{X}_{IIII} &= \tau(X_{ISII} + X_{IISI} + X_{IIIS}) + 3\tau X_{SIII} - 4\gamma X_{IIII}.
\end{aligned}
\right. \tag{*}
$$

This master equations can be rewritten as $\dot{X} = PX$ where

$$
P := \begin{pmatrix}
0 & \gamma & \gamma & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\tau-\gamma & 0 & 0 & 0 & \gamma & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\tau-\gamma & 0 & 0 & \gamma & 0 & 0 & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\tau-\gamma & 0 & 0 & \gamma & 0 & \gamma & 0 & \gamma & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -3\tau-\gamma & 0 & 0 & \gamma & 0 & \gamma & \gamma & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 & 0 & 0 & 0 & \gamma & \gamma & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 & 0 & 0 & \gamma & 0 & \gamma & 0 & 0 \\
0 & \tau & 0 & 0 & \tau & 0 & 0 & -2\tau-2\gamma & 0 & 0 & 0 & 0 & \gamma & \gamma & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 & \gamma & 0 & 0 & \gamma & 0 \\
0 & 0 & \tau & 0 & \tau & 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 & \gamma & 0 & \gamma & 0 \\
0 & 0 & 0 & \tau & \tau & 0 & 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 & \gamma & \gamma & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3\tau-3\gamma & 0 & 0 & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 2\tau & 0 & \tau & 0 & \tau & 0 & 0 & -\tau-3\gamma & 0 & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 2\tau & \tau & 0 & 0 & \tau & 0 & 0 & -\tau-3\gamma & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\tau & \tau & \tau & 0 & 0 & 0 & -\tau-3\gamma & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\tau & \tau & \tau & \tau & -4\gamma
\end{pmatrix}.
$$

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Let $B^0 = (0)$, $C^0 = (\gamma, \gamma, \gamma, \gamma)$,

$$A^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad B^1 = \begin{pmatrix} -\tau-\gamma & 0 & 0 & 0 \\ 0 & -\tau-\gamma & 0 & 0 \\ 0 & 0 & -\tau-\gamma & 0 \\ 0 & 0 & 0 & -\tau-\gamma \end{pmatrix}, \quad C^1 = \begin{pmatrix} \gamma & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & 0 & 0 & \gamma & \gamma & 0 \\ 0 & \gamma & 0 & \gamma & 0 & \gamma \\ 0 & 0 & \gamma & 0 & \gamma & \gamma \end{pmatrix},$$

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & \tau \\ 0 & 0 & \tau & \tau \end{pmatrix}, \quad B^2 = \begin{pmatrix} -2\tau-2\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\tau-2\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\tau-2\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\tau-2\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\tau-2\gamma \end{pmatrix}, \quad C^2 = \begin{pmatrix} \gamma & \gamma & 0 & 0 \\ \gamma & 0 & \gamma & 0 \\ 0 & \gamma & \gamma & 0 \\ \gamma & 0 & 0 & \gamma \\ 0 & \gamma & 0 & \gamma \\ 0 & 0 & \gamma & \gamma \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2\tau & 0 & \tau & 0 & \tau & 0 \\ 0 & 2\tau & \tau & 0 & 0 & \tau \\ 0 & 0 & 0 & 2\tau & \tau & \tau \end{pmatrix}, \quad B^3 = \begin{pmatrix} -3\tau-3\gamma & 0 & 0 & 0 \\ 0 & -\tau-3\gamma & 0 & 0 \\ 0 & 0 & -\tau-3\gamma & 0 \\ 0 & 0 & 0 & -\tau-3\gamma \end{pmatrix}, \quad C^3 = \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix},$$

$$A^4 = (3\tau, \tau, \tau, \tau), \quad B^4 = (-4\gamma),$$

then we obtain the block diagonal form of P;

$$P = \begin{pmatrix} B^0 & C^0 & & & O \\ A^1 & B^1 & C^1 & & \\ & A^2 & B^2 & C^2 & \\ & & A^3 & B^3 & C^3 \\ O & & & A^4 & B^4 \end{pmatrix}. \qquad \square$$