Report 1.

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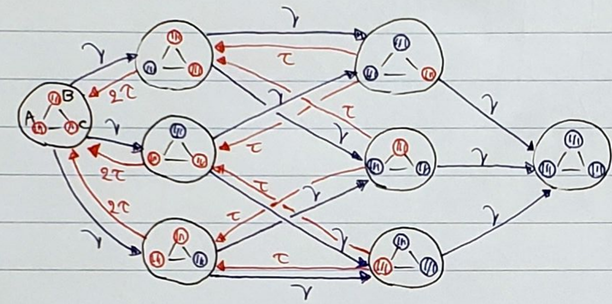
Exercise (Lecture note p. 20)

> Find a basis for the example of 3 nodes such that the transition matrix is block diagonal. Write this matrix explicitly.

∘ **Recall.** (Example. Lecture note p. 16)



S : susceptible
I : infected.
γ : recovery rate
τ : transmission rate.

Set X_{ABC} = " prob. that the system is in state ABC.
(S or I)

From the above figure, we have the following master equations;

$$
(*)\quad
\begin{cases}
\dot{X}_{SSS} = \gamma X_{SSI} + \gamma X_{SIS} + \gamma X_{ISS} \\
\dot{X}_{SSI} = \gamma X_{SII} + \gamma X_{ISI} - \gamma X_{SSI} - 2\tau X_{SSI} \\
\dot{X}_{SIS} = \gamma X_{SII} + \gamma X_{IIS} - 2\tau X_{SIS} - \gamma X_{SIS} \\
\dot{X}_{ISS} = \gamma X_{ISI} + \gamma X_{IIS} - 2\tau X_{ISS} - \gamma X_{ISS} \\
\dot{X}_{SII} = \gamma X_{III} + \tau X_{SSI} + \tau X_{SIS} - 2\tau X_{SII} - 2\gamma X_{SII} \\
\dot{X}_{ISI} = \gamma X_{III} + \tau X_{SSI} + \tau X_{ISS} - 2\tau X_{ISI} - 2\gamma X_{ISI} \\
\dot{X}_{IIS} = \gamma X_{III} + \tau X_{SIS} + \tau X_{ISS} - 2\tau X_{IIS} - 2\gamma X_{IIS} \\
\dot{X}_{III} = 2\tau X_{SII} + 2\tau X_{ISI} + 2\tau X_{IIS} - 3\gamma X_{III}.
\end{cases}
$$

Set $X := {}^{t}(X_{SSS}, X_{SSI}, X_{SIS}, X_{ISS}, X_{SII}, X_{ISI}, X_{IIS}, X_{III})$, (transpose)

Then one can rewrite (*) to the following;

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$$\dot{X} = PX, \quad \text{where}$$

$$P = \begin{pmatrix} 0 & \gamma & \gamma & \gamma & 0 & 0 & 0 & 0 \\ 0 & -2\tau-\gamma & 0 & 0 & \gamma & \gamma & 0 & 0 \\ 0 & 0 & -2\tau-\gamma & 0 & \gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & -2\tau-\gamma & 0 & \gamma & \gamma & 0 \\ 0 & \tau & \tau & 0 & -2\tau-2\gamma & 0 & 0 & \gamma \\ 0 & \tau & 0 & \tau & 0 & -2\tau-2\gamma & 0 & \gamma \\ 0 & 0 & \tau & \tau & 0 & 0 & -2\tau-2\gamma & \gamma \\ 0 & 0 & 0 & 0 & 2\tau & 2\tau & 2\tau & -3\gamma \end{pmatrix}.$$

Thus, putting $B^0 = (0)$, $C^0 = (\gamma, \gamma, \gamma)$, $A^1 = {}^t(0, 0, 0)$

$$B^1 = \begin{pmatrix} -2\tau-\gamma & 0 & 0 \\ 0 & -2\tau-\gamma & 0 \\ 0 & 0 & -2\tau-\gamma \end{pmatrix}, \quad C^1 = \begin{pmatrix} \gamma & \gamma & 0 \\ \gamma & 0 & \gamma \\ 0 & \gamma & \gamma \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \tau & \tau & 0 \\ \tau & 0 & \tau \\ 0 & \tau & \tau \end{pmatrix}, \quad B^2 = \begin{pmatrix} -2\tau-2\gamma & 0 & 0 \\ 0 & -2\tau-2\gamma & 0 \\ 0 & 0 & -2\tau-2\gamma \end{pmatrix}, \quad C^2 = {}^t(\gamma, \gamma, \gamma)$$

$$A^3 = (2\tau, 2\tau, 2\tau), \quad B^3 = (-3\gamma),$$

these are desired basis for this example. □

Exercise 2.4 ([KMS] p.37)

Do the same for a line graph with 3 nodes.

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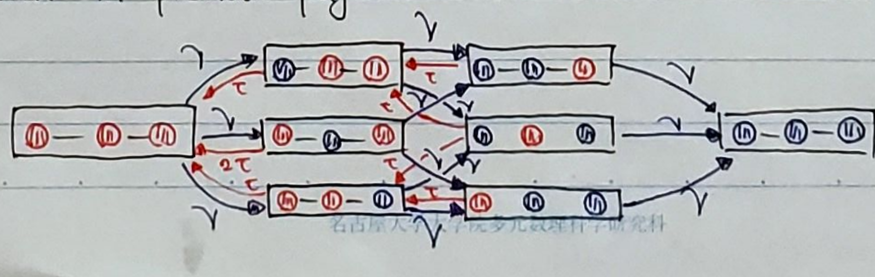
The all possible states are as follows. (we use the same notation in previous page.)



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So one has following master equations;

$$(*)\quad \begin{cases} \dot{X}_{SSS} = \gamma X_{SSI} + \gamma X_{SIS} + \gamma X_{ISS}, \\ \dot{X}_{SSI} = \gamma X_{SII} + \gamma X_{ISI} - (\tau+\gamma) X_{SSI} \\ \dot{X}_{SIS} = \gamma X_{SII} + \gamma X_{IIS} - (2\tau+\gamma) X_{SIS} \\ \dot{X}_{ISS} = \gamma X_{ISI} + \gamma X_{IIS} - (\tau+\gamma) X_{SSI} \\ \dot{X}_{SII} = \gamma X_{III} + \tau X_{SSI} + \tau X_{SIS} - (\tau+2\gamma) X_{SII} \\ \dot{X}_{ISI} = \gamma X_{III} - (2\tau+2\gamma) X_{ISI} \\ \dot{X}_{IIS} = \gamma X_{III} + \tau X_{SIS} + \tau X_{ISS} - (\tau+2\gamma) X_{IIS} \\ \dot{X}_{III} = -3\gamma X_{III} + \tau X_{SII} + \tau X_{IIS} + 2\tau X_{ISI} \end{cases}.$$

And we have $\dot{X} = PX$ where

$$P = \begin{pmatrix} 0 & \gamma & \gamma & \gamma & 0 & 0 & 0 & 0 \\ 0 & -\tau-\gamma & 0 & 0 & \gamma & \gamma & 0 & 0 \\ 0 & 0 & -2\tau-\gamma & 0 & \gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & -\tau-\gamma & 0 & \gamma & \gamma & 0 \\ 0 & \tau & \tau & 0 & -\tau-2\gamma & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 & 0 & -2\tau-2\gamma & 0 & \gamma \\ 0 & 0 & \tau & \tau & 0 & 0 & -\tau-2\gamma & \gamma \\ 0 & 0 & 0 & 0 & \tau & 2\tau & \tau & -3\gamma \end{pmatrix}.$$

Moreover, put $B^0 = (0)$, $C^0 = (\gamma, \gamma, \gamma)$, $A^1 = {}^t(0,0,0)$,

$$B^1 = \begin{pmatrix} -\tau-\gamma & 0 & 0 \\ 0 & -2\tau-\gamma & 0 \\ 0 & 0 & -\tau-\gamma \end{pmatrix}, \quad C^1 = \begin{pmatrix} \gamma & \gamma & 0 \\ \gamma & 0 & \gamma \\ 0 & \gamma & \gamma \end{pmatrix}, \quad A^2 = \begin{pmatrix} \tau & \tau & 0 \\ 0 & 0 & 0 \\ 0 & \tau & \tau \end{pmatrix},$$

$$B^2 = \begin{pmatrix} -\tau-2\gamma & 0 & 0 \\ 0 & -2\tau-2\gamma & 0 \\ 0 & 0 & -\tau-2\gamma \end{pmatrix}, \quad C^2 = {}^t(\gamma, \gamma, \gamma), \quad A^3 = (\tau, 2\tau, \tau).$$

$B^3 = (-3\gamma)$, one gets the block diagonal form of P;

$$P = \begin{pmatrix} B^0 & C^0 & 0 & 0 \\ A^1 & B^1 & C^1 & 0 \\ 0 & A^2 & B^2 & C^2 \\ 0 & 0 & A^3 & B^2 \end{pmatrix} \qquad \square$$