

Solution to the Exercise 1.52, at P.15, [GW]

Exercise 1.52

Here are two routine problems about balls in urns. You are presented with two urns. Urn I contains 3 white and 4 black balls, and Urn II contains 2 white and 6 black balls.

(a) You pick a ball randomly from Urn I and place it in Urn II. Next you pick a ball randomly from Urn II. What is the probability that the ball is black?

(b) This time, you pick an urn at random, each of the two urns being picked with probability $\frac{1}{2}$, and you pick a ball at random from the chosen urn. Given the ball is black, what is the probability you picked Urn I?

(a) $\Omega = (\text{2nd pick} = B)$

$$\Omega = \left\{ \begin{array}{c} \text{1st pick} = W \\ \uparrow \\ M_1 \end{array} \right\} \cup \left\{ \begin{array}{c} \text{1st pick} = B \\ \uparrow \\ M_2 = \emptyset \end{array} \right\} \quad (W = \text{white}, B = \text{black})$$

(i) After I pick a white ball from Urn I, there are 3 white balls and 6 black balls in Urn II.

The probability that I pick a white ball from Urn I is

$$P(M_1) = \frac{3}{7}$$

Therefore, the probability that I pick a black ball from Urn II, given that I picked a white ball from Urn I to Urn II, is

$$P(\text{2nd pick} = B | M_1) P(M_1) = \frac{6}{9} + \frac{3}{7} = \frac{18}{63}$$

(ii) After I pick a black ball from Urn I, there are 2 white balls and 7 black balls in Urn II.

The probability that I pick a black ball from Urn I is

$$P(M_2) = \frac{4}{7}$$

Therefore, the probability that I pick a black ball from Urn II, given that I picked a black ball from Urn I to Urn II, is

$$P(\text{2nd pick} = B | M_2) P(M_2) = \frac{7}{9} \times \frac{4}{7} = \frac{28}{63}$$

Thus, the probability that the ball is black is

$$\begin{aligned} P(\text{2nd pick} = B) &= P(\text{2nd pick} = B | M_1) P(M_1) + P(\text{2nd pick} = B | M_2) P(M_2) \\ &= \frac{18}{63} + \frac{28}{63} = \frac{46}{63} \end{aligned}$$

(b) According to the Bayes theorem, we have

$$P(\text{picked Urn I} | \text{picked B})$$

$$= \frac{P(\text{picked B} | \text{picked Urn I}) P(\text{picked Urn I})}{P(\text{picked B} | \text{picked Urn I}) P(\text{picked Urn I}) + P(\text{picked B} | \text{picked Urn II}) P(\text{picked Urn II})}$$

$$= \frac{\frac{4}{9} \times \frac{1}{2}}{\frac{4}{9} \times \frac{1}{2} + \frac{6}{9} \times \frac{1}{2}} = \frac{16}{37}$$