SML: Probability (特別数学講義：確率論) 14th April Lecture

Let $\{B_i\}_{i \in I}$ be a partition of Ω (the sample space)

Let $A \in \mathcal{F}$ (an event space of the sample space Ω).

Statement to be proven: $\sum_{i \in I} \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i) = \mathbb{P}(A)$

By definition of $\mathbb{P}(A \mid B)$, one has:

$$\sum_{i \in I} \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i) = \sum_{i \in I} \mathbb{P}(A \cap B_i)$$

Since $\{B_i\}_{i \in I}$ is pairwise disjoint, ($\{B_i\}_{i \in I}$ is a partition),

one has that $\{A \cap B_i\}_{i \in I}$ is also pairwise disjoint.

As such, $\sum_{i \in I} \mathbb{P}(A \cap B_i) = \mathbb{P}\left(\bigcup_{i \in I} (A \cap B_i)\right)$

Claim: $\bigcup_{i \in I} (A \cap B_i) = A \cap \left(\bigcup_{i \in I} B_i\right)$

For the elementary event ω, $\omega \in \bigcup_{i \in I} (A \cap B_i)$ implies $\omega \in A$.
Also, $\omega \in \bigcup_{i \in I} (A \cap B_i)$ implies that exists $i \in I$ such that $\omega \in B_i$.
Therefore, for this i, $\omega \in A$ and $\omega \in \bigcup_{i \in I} B_i$, thus, $\omega \in A \cap \left(\bigcup_{i \in I} B_i\right)$.

For the elementary event ω, $\omega \in A \cap \left(\bigcup_{i \in I} B_i\right)$ implies $\omega \in A$.
$\omega \in A \cap \bigcup_{i \in I} B_i$ also implies that exists $i \in I$ such that $\omega \in B_i$.
Therefore, for this i, $\omega \in A \cap B_i$, which means that $\omega \in \bigcup_{i \in I} (A \cap B_i)$

As such, one concludes that LHS = RHS (claim proven).

Following the claim above, one can observe that:

$$\bigcup_{i \in I} (A \cap B_i) = A \cap \left(\bigcup_{i \in I} B_i\right) \underset{\{B_i\}_{i \in I} \text{ is a partition}}{=} A \cap \Omega \overset{A \in \Omega}{=} A \quad \text{.... [1]}$$

$$\text{As such, } \sum_{i \in I} \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i) = \mathbb{P}\left(A \cap \left(\bigcup_{i \in I} B_i\right)\right) \overset{\text{by [1]}}{=} \mathbb{P}(A) \quad \text{(Statement Proven.)}$$