A Problem about Probability Generating Functions

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Question

Each year a tree of a particular type flowers once, and the probability that it has n flowers is $(1-p)p^n$, $n = 0, 1, 2...$, where $0 < p < 1$. Each flower has probability $\frac{1}{2}$ of producing a ripe fruit, independently of all other flowers. We assume that each flower produces only 1 ripe fruit in maximum. Find the probability that in a given year:

(a) the tree produces r ripe fruits,
(b) the tree had n flowers, given that it produces r ripe fruits.

Solution

(a) Given that the tree produces $N = n$ flowers, and let the number of ripe fruits be R. We know the number of ripe fruits follows a binomial distribution: binomial$(n, \frac{1}{2})$.

One can compute the probability generating function

$$\begin{aligned}
G_R(s) &= \sum_{r=0}^{\infty} \mathbb{P}(R = r)s^r = \sum_{r=0}^{\infty}\sum_{n=r}^{\infty} \mathbb{P}(R = r|N = n)\mathbb{P}(N = n)s^r \\
&= \sum_{n=0}^{\infty} \mathbb{P}(N = n)\sum_{r=0}^{n} \mathbb{P}(R = r|N = n)s^r \\
&= \sum_{n=0}^{\infty}(1-p)p^n(\frac{1+s}{2})^n = \frac{1-p}{1-p\frac{1+s}{2}} = \frac{2-2p}{2-p-ps} \\
&= \sum_{r=0}^{\infty} \frac{2-2p}{2-p}(\frac{p}{2-p})^r s^r.
\end{aligned}$$

Then, one can readily conclude the probability that the tree produces r ripe fruits is

$$\mathbb{P}(R = r) = \frac{2-2p}{2-p}(\frac{p}{2-p})^r. \quad \square$$

(b) Applying the definition of conditional probabilities:

$$\mathbb{P}(A|B)P(B) = \mathbb{P}(A \cap B) = \mathbb{P}(B \cap A) = \mathbb{P}(B|A)\mathbb{P}(A). \tag{1}$$

$$\Longrightarrow \mathbb{P}(B|A) = \mathbb{P}(A|B)\frac{\mathbb{P}(B)}{P(A)}. \tag{2}$$

From the equality (2), one can compute the probability that the tree had n flowers, given that it produces r ripe fruits is

$$\begin{aligned}\mathbb{P}(N=n|R=r) &= \frac{\mathbb{P}(R=r|N=n)\mathbb{P}(N=n)}{\mathbb{P}(R=r)}, \\ &= \frac{\binom{n}{r}\frac{1}{2^n}(1-p)p^n}{\frac{2-2p}{2-p}(\frac{p}{2-p})^r}, \\ &= \frac{\binom{n}{r}p^{n-r}(2-p)^{r+1}}{2^{n+1}},\end{aligned}$$

where $0 \leq r \leq n$. $\square$